Mathematics: Analysis and Approaches HL

Chapter summaries

Haese Mathematics

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CHAPTER 1: FURTHER TRIGONOMETRY

- **A** Reciprocal trigonometric functions
- **B** Inverse trigonometric functions
- **C** Algebra with trigonometric functions
- **D** Double angle identities
- **E** Compound angle identities

Syllabus reference: SL 3.6, SL 3.8, AHL 3.9, AHL 3.10, AHL 3.11

This chapter follows directly from the final chapter of the Mathematics: Core Topics HL book. It contains the remaining material on trigonometry and trigonometric functions that is only for Analysis and Approaches HL students.

When graphing reciprocal trigonometric functions, students should be reminded of their work in graphing $y = \frac{1}{f(x)}$ in Chapter 16 of the Mathematics: Core Topics HL book. This will help them to sketch the graph of $y = \sec x$ from the graph of $y = \cos x$.

Care should be taken to consider the domain and range of inverse trigonometric functions. It is important to realise that we only consider the functions $y = \sin x$, $\cos x$, and $\tan x$ over a restricted domain so that the inverse function exists. The domain and range of these functions are also important when solving equations involving them.

Students must also use trigonometric identities to simplify expressions and solve equations. It would be beneficial for students to understand how identities are derived, rather than trying to memorise them. For example, they should understand that $\cos\theta = \cos(-\theta)$ can be established by a reflection of points on the unit circle in the x-axis, and that the Pythagorean identity can be used to find the equivalent expressions for $\cos 2\theta$.

The trigonometric series and product identities that were in this chapter in previous editions have been pushed back to Chapter 10 (Proof by mathematical induction).

CHAPTER 2: EXPONENTIAL FUNCTIONS

- **A** Rational exponents
- **B** Algebraic expansion and factorisation
- **C** Exponential equations
- **D** Exponential functions
- **E** Growth and decay
- **F** The natural exponential

Syllabus reference: SL 1.5, SL 1.7, SL 2.9, SL 2.10

We begin the chapter with the study of rational exponents. This gives meaning to the value of a^x for all rational x, and lays the foundation for the study of exponential functions, which define a^x for all real x.

Students solve exponential equations by equating exponents. It should be pointed out to students that this can only be done for particular cases of exponential equations, where it is easy to write both sides of the equation in the same base. This will set the scene for being able to solve a wider range of exponential equations using logarithms in the next chapter.

Students will use the transformations studied in Chapter 16 of the Mathematics: Core Topics HL book to predict the shape of graphs such as $y = 2^x + 3$ and $y = 3^{x-1} - 4$.

In the Discussion at the end of Section D, students are asked why we specify a positive base number a. As a starting point, they should recognise that, since $a^{\frac{1}{2}} = \sqrt{a}$, a^x will be undefined at $x = \frac{1}{2}$ for negative values of a. By considering $y = (-2)^x$ for the integer values of x, students should realise that essentially two subgraphs start to form: the graph of $y = 2^x$ for even values of x, and $y = -(2^x)$ for odd values of x. When considering its domain, students should find that the function is undefined for all rational x with even denominator. Students may find the graphing package useful for exploring functions of this type.

CHAPTER 3: LOGARITHMS

A Logarithms in base 10
B Logarithms in base a
C Laws of logarithms
D Natural logarithms
E Logarithmic equations
F The change of base rule
G Solving exponential equations using logarithms
H Logarithmic functions

Syllabus reference: SL 1.5, SL 1.7, SL 2.9

At the very start of the chapter, we introduce logarithmic functions as the inverse of exponential functions, to motivate their use. We then proceed through the standard properties of logarithms, before returning to logarithmic functions at the end of the chapter.

In Section G, students learn how to use logarithms to solve any exponential equation. Where necessary, we use the logarithm laws to write the solution in terms of base 10 logarithms, which can then be approximated using a calculator. The growth and decay questions first posed in Chapter 2 are now extended to answer questions like "how long will it take for the population to reach 500?".

CHAPTER 4: INTRODUCTION TO COMPLEX NUMBERS

A Complex numbers
B The sum of two squares factorisation
C Operations with complex numbers
D Equality of complex numbers
E Properties of complex conjugates

Syllabus reference: AHL 1.12, AHL 1.13, AHL 1.14

In this latest edition, we have decided to split up complex numbers and polynomials into their own chapters. Each is conceptually deep, and sufficient in length to warrant a chapter of their own. It also helps to better delineate the aspects of quadratic functions which refer to complex numbers, and the study of polynomials.

In this introductory chapter to complex numbers, we consider the algebra of complex numbers. The more advanced work involving the geometry of complex numbers will be covered in a later chapter.

The Opening Problem and Historical Note at the start of the chapter aim to lead students to the idea that it is *useful* to define the square root of a negative number. Students should see that, once we define $i = \sqrt{-1}$, we can find the solutions to *any* quadratic equation.

It is important that students recognise the analogy between using radical conjugates to perform division with radicals, and using complex conjugates to perform division with complex numbers.

CHAPTER 5: REAL POLYNOMIALS

A Polynomials
B Operations with polynomials
C Zeros, roots, and factors
D Polynomial equality
E Polynomial division
F The Remainder theorem
G The Factor theorem
H The Fundamental Theorem of Algebra
I Sum and product of roots theorem
J Graphing cubic functions
K Graphing quartic functions
L Polynomial equations
M Cubic inequalities

Syllabus reference: SL 2.10, AHL 2.12, AHL 2.15

We now consider the properties, operations, theorems, and graphs associated with real polynomials.

Now that complex numbers have been introduced in the previous chapter, we are in a position to paint a complete picture about the zeros of real polynomials. In Section C, students are asked to discuss whether a zero of a polynomial will always correspond to an x-intercept of its graph. Students should conclude that this is only the case for real zeros. For example, the polynomial $P(x) = x^2 + 1$ has the complex zeros i and $-i$, but these zeros do not correspond to x-intercepts of the graph of the function.

The notion of real and complex zeros culminates in Section H (The Fundamental Theorem of Algebra), in which we find that every real polynomial of degree n has exactly n zeros.

In Section I (Sum and product of roots theorem), students should be encouraged to see this result as a more general case of the result found in Chapter 4 of the Mathematics: Core Topics HL book, for the sum and product of the roots of quadratic equations.

The Discussion at the end of Section K invites students to describe the behaviour of a general polynomial function. Students should find that polynomials of even degree approach ∞ as x approaches $\pm\infty$ if a_n is positive, and approach $-\infty$ as x approaches $\pm\infty$ if a_n is negative. Polynomials of odd degree head in different directions as x aproaches ∞ and as x approaches $-\infty$. This should lead students to conclude that polynomials of odd degree *must* have at least one real zero, as the graph of the polynomial must cross the x-axis.

When solving cubic equations and inequalities, we use technology to identify a rational zero of the polynomial, and then use this information to determine the remaining zeros.

CHAPTER 6: FURTHER FUNCTIONS

A Even and odd functions
B The graph of $y = [f(x)]^2$
C Absolute value functions
D Rational functions
E Partial fractions

Syllabus reference: AHL 2.13, AHL 2.14, AHL 2.16, AHL 1.11

We have placed this functions chapter after polynomials so that we can use the techniques of polynomial division in our study of rational functions. This also allows us to use polynomials in our exploration of even and odd functions.

In Section A we explore whether the trigonometric functions are even or odd, and discuss the geometric interpretation of these results.

In Section B, students study the graph of $y = [f(x)]^2$, as outlined in the syllabus. We find this to be an unusual aspect of functions to study, but probably the most insightful approach is to concentrate on the behaviour of the function at its x-intercepts. Since all of the linear factors of the function become quadratic functions, the graph of $y = [f(x)]^2$ *touches* the x-axis where the graph of $y = f(x)$ *cuts* the x-axis.

We have not included the graph of $f(ax-b)$ in this chapter, since it is covered in Chapter 16 of the Mathematics: Core Topics HL book.

In Section C, we consider graphs of the form $y = |f(x)|$ and $y = f(|x|)$, as well as solve modulus equations and inequalities, both algebraically and graphically.

At the end of Section D, students must discuss the factors which determine the shape of the rational function $y = \dfrac{ax^2+bx+c}{dx+e}$. It would be useful to direct students back to the graphs they have studied of this form in question **2** of Exercise 6D.2. In particular, they should consider the functions written in the form $y = px + q + \dfrac{r}{dx+e}$, and think about the effect of the sign of r.

Section E (Partial fractions) is placed here because it will allow students to extend on their work with rational functions, to write them as the sum of partial fractions. However, this work will be most useful in calculus, later in the course.

CHAPTER 7: COUNTING

- **A** The product principle
- **B** The sum principle
- **C** Factorial notation
- **D** Permutations
- **E** Combinations

Syllabus reference: SL 1.9, AHL 1.10

In previous editions of Mathematics HL, we have placed Counting and the binomial expansion together in a single chapter. We have decided to place them in separate chapters in this book. While the topics are clearly linked, we believe that each concept is deep and important enough to exist as a chapter in its own right. Also, the extension of the binomial theorem to include rational exponents bulks up this material, and it is therefore more sensible to split the material into two chapters.

There is a Discussion regarding the definition of $0! = 1$. Students should find that this definition, whilst it is meaningless in the context of the original definition of factorial, is not "arbitrary", in that it was sensibly chosen in order that properties of factorial, such as $n! = n \times (n-1)!$, still apply. When considering other areas of mathematics where definitions are expanded in this way, students could discuss how the definition of exponents is expanded to include zero, negative, and rational exponents, or how the definition of trigonometric ratios is expanded to include angles larger than 90 degrees. This is also a useful thing to keep in mind in the next chapter, where the definition of ${}^nC_r$ is extended to negative and rational n.

When introducing permutations with some objects repeated, it may help to explain that we divide by, say, $n_1!$ because the order in which the n_1 copies of the first symbol appear is not important, as they are identical. This leads well into the study of combinations, and the motivation of the formula ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, since the ordering of the r selected items is not important, *and* the ordering of the $(n-r)$ unselected items is not important.

CHAPTER 8: THE BINOMIAL THEOREM

- **A** Binomial expansions
- **B** The binomial theorem for $n \in \mathbb{Z}^+$
- **C** The binomial theorem for $n \in \mathbb{Q}$

Syllabus reference: SL 1.9, AHL 1.10

The material in Section C has not been in HL Mathematics courses before this course change, so it is likely to be a challenge for both students and teachers.

The first important point to notice is that we must extend the definition of the binomial coefficient $\binom{n}{r}$. Students should understand that this definition is consistent with the original definition for positive integers n, but can also be applied to negative or rational values of n.

An important distinction students should recognise between the binomial theorem for positive integer powers, and for rational powers, is that the binomial expansion for rational powers contains an infinite series of terms. Therefore, we generally only find the first few terms of the expansion.

Students should also understand that the binomial expansion $(a + bx)^n$ where $n \in \mathbb{Q}$ will only converge for certain values of x. While a formal proof of this is not required for the course, it will be helpful to draw the analogy with infinite geometric sequences, which also contain an infinite series of terms, and which only converge when the common ratio r satisfies $-1 < r < 1$.

This expansion will be returned to in Chapter 24, when we study Maclaurin series. It is hoped that, for students who do not completely understand this material here, it will make more sense when the connection with Maclaurin series is made.

CHAPTER 9: REASONING AND PROOF

- **A** Logical connectives
- **B** Proof by deduction
- **C** Proof by equivalence
- **D** Definitions
- **E** Proof by exhaustion
- **F** Disproof by counter example
- **G** Proof by contrapositive
- **H** Proof by contradiction: reductio ad absurdum

Syllabus reference: SL 1.6, AHL 1.15

Logical connectives are discussed as they are necessary to understand how to construct a sound argument, with one statement leading to the next. The concept of definitions is important as you need them as basis of a proof, and without them, you cannot really prove anything.

We have steered clear of "geometric style" proofs, as these are not mentioned in the syllabus, and students are likely to have encountered them in previous years.

We have distinguished between "proof by implication", in which only the correct implication from one step to the next is required, and "proof by equivalence", in which each deductive step must be an equivalence, allowing us to establish the equivalence of two statements.

In SL 1.6 of the syllabus, it is stated that the expressions $\frac{1}{m+1} + \frac{1}{m^2+m}$ and $\frac{1}{m}$ are equivalent. However, this is not the case, as when $m = -1$ the LHS is undefined, whereas the RHS is defined. We have included a question in Exercise 9C to address this.

We have included proof by contrapositive, as this is a form of direct proof. The chapter ends with proof by contradiction, which is a form of indirect proof.

In a Discussion at the end of Section H, students should recognise the distinction between labelling proofs as "direct" and "indirect", and labels like "proof by equivalence", and so on. The Discussion also asks why it is so hard to prove that π is irrational. As part of their answer, students should consider why it is easier to prove a number is rational then to prove it is irrational. They should also consider why proving $\sqrt{2}$ is irrational is easier than proving π is irrational.

The final Discussion of the chapter asks students to discuss what features of a problem point towards using particular forms of proof. Students should consider that more algebraic style proofs, for example that two expressions are equal, should be solved using proof by equivalence. Proofs which require a statement to be proved for all integers n, say, may be proved by exhaustion if we can sensibly divide the integers into classes (such as odd and even integers). Proving that a number is irrational, or that there are infinitely many numbers with a particular property, is difficult to do directly, so an indirect proof such as proof by contradiction may be more appropriate.

CHAPTER 10: PROOF BY MATHEMATICAL INDUCTION

- **A** The process of induction
- **B** The principle of mathematical induction

Syllabus reference: AHL 1.15

Proof by mathematical induction is given a chapter of its own, as the previous chapter is already quite long and conceptually demanding, and induction can be applied to many different topics. As we have already covered trigonometry, the induction proofs about trigonometric series and products are now in this chapter, rather than the trigonometry chapter as in previous editions. Induction questions will also appear in later chapters where appropriate, in topics such as calculus and statistics.

The chapter ends with a wonderful Activity about infinite descent, and students who finish the chapter ahead of their classmates should be encouraged to have a look at this.

We would expect that classes should complete this chapter by the end of the first year of studies.

CHAPTER 11: LINEAR ALGEBRA

- **A** Systems of linear equations
- **B** Row operations
- **C** Solving 2×2 systems of linear equations
- **D** Solving 3×3 systems of linear equations

Syllabus reference: AHL 1.16

Students are asked to solve 2×2 systems of equations using row operations, even though these systems have previously been solved using substitution or elimination. This is done to familiarise the students with the process in a more simple context, so they are comfortable performing row operations when they come to 3×3 systems, where it is most useful.

In Section D, we have been intentionally prescriptive about the forms the students may obtain when reducing the matrix, and the interpretation of each. We feel this is clearer than trying to explain the numerous possible outcomes in terms of a general matrix.

The work in this chapter forms a solid basis for finding the intersection of planes later in the course.

We have left solution by technology until the end, so as not to interrupt the flow of ideas.

CHAPTER 12: VECTORS

- **A** Vectors and scalars
- **B** Geometric operations with vectors
- **C** Vectors in the plane
- **D** The magnitude of a vector
- **E** Operations with plane vectors
- **F** Vectors in space
- **G** Operations with vectors in space
- **H** Vector algebra
- **I** The vector between two points
- **J** Parallelism
- **K** The scalar product of two vectors
- **L** The angle between two vectors
- **M** Proof using vector geometry
- **N** The vector product of two vectors

Syllabus reference: AHL 3.12, AHL 3.13, AHL 3.16

This chapter is quite long, and we have described vectors from scratch as we cannot guarantee that students have seen vectors before. However, if your students have seen vectors before, it may be sensible to move quickly through the first few sections of this chapter.

In the Discussion at the end of Section A, students should conclude that a vector of length 0 does not have direction, and such a vector can only be represented geometrically as a point, rather than a line segment.

The intention of the Discussion at the end of Section B is to get students to think about the limitations of representing vectors geometrically. This is a good lead-in to the following section, which introduces vectors in component form.

Once we consider 3-dimensional vectors in Section F, it is no longer practical to represent the vectors geometrically, and so almost all of the work is done in component form. Students should see how the properties of 3-dimensional vectors extend readily from those of 2-dimensional vectors. From Section H onwards, 2-dimensional and 3-dimensional vectors are presented together.

Care should be taken to distinguish between the modulus of a scalar, and the length of a vector.

In Section N, to help students remember the formula for the vector product, we have included an Investigation which defines the determinant of a matrix, and explores how the vector product formula can be written in terms of determinants.

CHAPTER 13: VECTOR APPLICATIONS

- **A** Lines in 2 and 3 dimensions
- **B** The angle between two lines
- **C** Constant velocity problems
- **D** The shortest distance from a point to a line
- **E** Intersecting lines
- **F** Relationships between lines
- **G** Planes
- **H** Angles in space
- **I** Intersecting planes

Syllabus reference: AHL 3.14, AHL 3.15, AHL 3.17, AHL 3.18

In this chapter we use vectors to explore lines, planes, and the relationship between them.

A key difference between lines and vectors that students should keep in mind is that lines do not have inherent "direction" to them. This has important consequences when considering the uniqueness of the equation of a line, and the angle between two lines.

In the Discussion in Section A, students should find that it does not make sense to talk about the gradient of a line in space.

In Section E, students must solve equations simultaneously to find the intersection of two lines. In Section F we use row reduction to solve the equations, as this helps us to discern the relationship between the lines when there is not a unique solution. Again, we do not necessarily need to use row reduction for these 2×2 systems, but it is a good introduction to solving 3×3 systems for intersecting planes in Section I.

In Section I, we extend the row reduction methods first studied in Chapter 11, to give a geometric interpretation of the situation. Students should understand that, when finding where planes intersect, the solution by row reduction only tells you whether there is a unique solution, no solution, or infinitely many solutions. To give a geometric interpretation of what is happening, you need to look beyond the reduced matrix, and reconsider the original equations of the plane.

CHAPTER 14: COMPLEX NUMBERS

- **A** The complex plane
- **B** Modulus and argument
- **C** Geometry in the complex plane
- **D** Polar form
- **E** Euler's form
- **F** De Moivre's theorem
- **G** Roots of complex numbers

Syllabus reference: AHL 1.12, AHL 1.13, AHL 1.14

The Opening Problem of this chapter extends from the Opening Problem of Chapter 4, and invites students to consider how complex numbers could be represented geometrically.

It would be worthwhile asking students to examine the solution to Example 3 at the end of Section A. They should see that, when represented in Cartesian form, there appears to be no connection between the complex numbers z and w, and their product zw. This will further emphasise the usefulness of polar form in multiplying complex numbers.

The power of polar form is shown to even greater effect in Sections E and F, where we find the powers and roots of complex numbers. It would be useful to remind students of their previous work in polynomials, where it was found that polynomials of degree n have exactly n zeros. When we find the nth roots of a complex number c, we are finding the zeros of $z^n - c$. Thus there are exactly n nth roots of c.

CHAPTER 15: LIMITS

- **A** Limits
- **B** The existence of limits
- **C** Limits at infinity
- **D** Trigonometric limits
- **E** Continuity

Syllabus reference: SL 5.1, AHL 5.12

In the Mathematics: Analysis and Approaches SL book, there was only a brief discussion of limits, so it was introduced where it was needed in the introduction to differential calculus. In HL however, there is a much broader discussion of limits, including the existence of limits, trigonometric limits (which are useful for establishing later rules for differentiation), limits at infinity (which are used in integration), and continuity. For this reason, we have included a complete chapter dealing with limits.

CHAPTER 16: INTRODUCTION TO DIFFERENTIAL CALCULUS

- **A** Rates of change
- **B** Instantaneous rates of change
- **C** The gradient of a tangent
- **D** The derivative function
- **E** Differentiation from first principles
- **F** Differentiability and continuity

Syllabus reference: SL 5.1, AHL 5.12

This chapter provides students with their first look at differential calculus. Although some calculus content is common between the HL courses, we expect the classes will be separated by the time they encounter calculus. Having the calculus chapters in the separate books allows a more targeted approach to calculus for each course.

This chapter begins with average rates of change, which is used to motivate a study of instantaneous rates of change, leading to the idea of using limits to find the gradient of the tangent to a curve.

In the Discussion at the end of Section C, students are asked to list reasons why the limit required to find the gradient of a tangent at $x = a$, may not exist. This may be because the function itself is undefined at $x = a$, or the function may be undefined for $x < a$ or $x > a$. Also, $\dfrac{f(a+h) - f(a)}{h}$ may approach different values as $h \to 0$ from above and below.

In the final section, we return to the concept of continuity, but this time it is used as a tool to test for differentiability. At this stage only a limited range of functions are discussed, as derivatives can only be found by first principles.

CHAPTER 17: RULES OF DIFFERENTIATION

- **A** Simple rules of differentiation
- **B** The chain rule
- **C** The product rule
- **D** The quotient rule
- **E** Derivatives of exponential functions
- **F** Derivatives of logarithmic functions
- **G** Derivatives of trigonometric functions
- **H** Derivatives of inverse trigonometric functions
- **I** Second and higher derivatives
- **J** Implicit differentiation

Syllabus reference: SL 5.3, SL 5.6, SL 5.7, AHL 5.12, AHL 5.14, AHL 5.15

In this chapter students will discover rules for differentiating functions. We use limits at infinity to motivate the natural exponential function e^x as the function which is its own derivative. Many of the proofs of these rules, while not required, are included to aid in students' understanding.

In Section F, we find the derivative of $y = \ln x$. Students can click on a link to obtain a graphical proof for this rule, but a more straightforward proof will be established in Section J (Implicit differentiation).

In Sections G and H, we present rules for inverse and reciprocal trigonometric functions. The rules for reciprocal trigonometric functions in particular can be difficult to remember, and students should be encouraged to become familiar with the process of deriving these rules using the chain rule, so memorisation is not required. Students should also be reminded that these rules will be included in their examination formula booklet.

There are a large number of differentiation rules in this chapter. Students should not get too bogged down trying to remember the derivatives involving compositions, such as $e^{f(x)}$ or $\sin(f(x))$. If students understand how the chain rule works, then finding the derivative of $\sin(f(x))$ is straightforward as long as you can find the derivative of $\sin x$.

Throughout this chapter, students should be encouraged to remember what a derivative function means, rather than just performing the differentiation without giving thought to its meaning. To this end, we have included exercises asking students to answer questions involving the gradients of tangents to the functions they are differentiating.

CHAPTER 18: PROPERTIES OF CURVES

A Tangents
B Normals
C Increasing and decreasing
D Stationary points
E Shape
F Inflection points
G Understanding functions and their derivatives
H L'Hôpital's rule

Syllabus reference: SL 5.2, SL 5.4, SL 5.7, SL 5.8, AHL 5.13

This chapter allows students to apply the calculus they have learnt to discover the properties of curves.

In the previous Mathematics HL book, we dealt with tangents and normals together in a single section, however we have placed them in separate sections here as the tangents section is quite large, and there are sufficient concepts presented in this exercise to warrant a section of its own.

The theory of much of this chapter is very similar to that in the Mathematics: Analysis and Approaches SL book, but more complicated function types, as well as implicit differentiation, are included in the exercises.

Students should be encouraged to think of the concepts of increasing and decreasing in terms of intervals, rather than at a particular point. This will help students understand why, for example, the graph of $y = x^2$ is increasing for $x \geqslant 0$, and decreasing for $x \leqslant 0$.

In Section D we return to sign diagrams, and see how the sign diagram of $f'(x)$ can be used to determine the nature of the stationary points of $f(x)$.

We have added a section dealing with shape before we introduce inflection points. This is similar to how we talk about increasing and decreasing functions before we deal with stationary points.

L'Hôpital's rule is included in this chapter, and as such questions are presented in context of finding properties of functions, such as asymptotes.

CHAPTER 19: APPLICATIONS OF DIFFERENTIATION

A Rates of change
B Optimisation
C Related rates

Syllabus reference: SL 5.1, SL 5.8, AHL 5.14

In this final chapter of differential calculus, we explore some of its real world applications. The important skill in this chapter is to take the calculus techniques learnt in previous chapters, apply them to real life problems, and to interpret the results in the context of that problem.

Students should keep in mind the constraints imposed by the context of the problem, and to make sure their solution makes sense in this context. This is especially true in problems involving trigonometry, in which we must remember that, for example, $\sin x$ can only take values from -1 to 1.

Related rates may be a challenge for some students. A good approach to explain the change in mindset for these questions is that previously we have taken a variable y in terms of x, and simply differentiated by x. Now, because we are interested in how both of these variables interact over time, we differentiate the whole equation with respect to time t, so both variables are differentiated with respect to t. The final question in this exercise asks students to prove Snell's law of reflection, which provides a wonderful example of the usefulness of mathematics in establishing laws across other fields of study.

The chapter ends with an online Activity about cubic spline interpolation. This could potentially be a useful starting point for a student's Mathematical Exploration.

CHAPTER 20: INTRODUCTION TO INTEGRATION

- **A** Approximating the area under a curve
- **B** The Riemann integral
- **C** Antidifferentiation
- **D** The Fundamental Theorem of Calculus

Syllabus reference: SL 5.5, SL 5.11

We begin our study of integration by calculating the area under the curve, using the idea of limits. We feel this approach is consistent with how integral calculus was developed historically. Limits at infinity are used here to explore what happens as we consider more and more upper and lower rectangles.

The title of Section B may appear intimidating to students, but all that is happening here is that we are using the integral symbol as notation to denote the area under a curve. It will be beneficial for students to understand this notation before moving on to see how the area under a curve relates to the antiderivative of functions.

At the end of Section B, there is an interesting Investigation which follows Fermat's development of an exact formula for the area under $y = x^k$. Students who have finished the exercises early should be encouraged to explore this Investigation. Students should recognise that here we are finding an exact formula analytically, as opposed to approximating the area numerically as in the exercises.

We then move on to consider antiderivatives of functions, culminating in the Fundamental Theorem of Calculus, which links antiderivatives and the area under a curve.

This is a short chapter, but is quite involved conceptually, so it is important that students spend the time to understand the link between antiderivatives and the area under a curve.

CHAPTER 21: TECHNIQUES FOR INTEGRATION

- **A** Discovering integrals
- **B** Rules for integration
- **C** Particular values
- **D** Integrating $f(ax+b)$
- **E** Partial fractions
- **F** Integration by substitution
- **G** Integration by parts

Syllabus reference: SL 5.5, SL 5.10, AHL 5.15, AHL 5.16

Now that we have established that integration is the reverse process of differentiation, we use the rules for differentiation in reverse to develop the rules for integration. At this stage we only consider indefinite integration.

As with the rules of differentiation, there are many rules for integration in this chapter. Students should be familiar with the contents of the formula booklet, and recognise that these rules can be derived from differentiation, rather than trying to memorise them all.

In Section E, we use the techniques from Chapter 6 to write rational functions as the sum of partial fractions. This allows us to integrate rational functions.

Some students may find integration by substitution challenging, as it is not always obvious what substitution should be made. We have provided a short guide for substitutions which should be tried in certain circumstances. However, this guide is not exhaustive, and students should find that, with practice, they will develop an intuition for the type of substitution which should be used.

In some instances, extra information about the original function is included, allowing us to determine the constant of integration. Occasionally simultaneous equations will be required to find multiple unknowns.

CHAPTER 22: DEFINITE INTEGRALS

- **A** Definite integrals
- **B** Definite integrals involving substitution
- **C** The area under a curve
- **D** The area above a curve
- **E** The area between two functions
- **F** The area between a curve and the y-axis
- **G** Solids of revolution
- **H** Problem solving by integration
- **I** Improper integrals

Syllabus reference: SL 5.5, SL 5.11, AHL 5.17

Once we have established the rules for integration, we now have more tools to calculate definite integrals, and to explore the relationship between definite integrals and areas.

When the integration requires substitution, students must make sure to transform the endpoints in the definite integral as well.

Areas under and above curves are treated separately, giving students more of an opportunity to see that the definite integral is a *signed* area function. This approach also allows students to practice with areas above curves in a section in its own right, rather than as a special case of the area between two curves $f(x)$ and $g(x)$ where $f(x) = 0$.

The syllabus specifies that some definite integrals will not be able to be performed analytically, and so technology must be used. To this end, we have included calculator instructions, screenshots, and exercises which allow students to practise finding definite integrals using technology.

In Section F, we have given the formula for the area between a curve and the y-axis as $\displaystyle\int_c^d f^{-1}(y)\,dy$ rather than $\displaystyle\int_c^d x\,dy$ to better indicate what is required for the integration, and to emphasise that the function f must be invertible. In the Discussion at the end of the section, students should conclude that f must be invertible so that we can write x as a function of y. To find the shaded area shown in the Discussion, students should think about the area under the curve $y = f(x)$, as well as the area bd of the large rectangle, and the area ac of the small rectangle.

Section H ends with an Activity about Buffon's needle problem. Some of the integration involved, particularly obtaining the probability given in **3 b** of the long needle case, is quite difficult, so it should not cause students concern if they cannot do this. However, it is hoped that the more able student will find this Activity interesting and engaging.

It is not clear whether improper integration is included in the syllabus, but given students have studied limits at infinity, it is a fairly intuitive next step. However, classes that are short on time may choose to omit this section. A Discussion in this section asks students why $\displaystyle\lim_{x\to\infty} f(x) = 0$ if $\displaystyle\int_a^\infty f(x)\,dx$ is to exist. Students should find that if $\displaystyle\lim_{x\to\infty} f(x) \neq 0$, the area between $f(x)$ and the x-axis must be infinite.

CHAPTER 23: KINEMATICS

- **A** Displacement
- **B** Velocity
- **C** Acceleration
- **D** Speed

Syllabus reference: SL 5.9

Having dealt with both differentiation and integration, we will now bring these processes together in the study of kinematics. Rather than dealing first with differentiation, then with integration, we will deal with particular concepts of kinematics, and present the differentiation and integration aspects of each concept together.

Most students are likely to have previously encountered questions concerning motion in the form of travel graphs. A difficulty students are likely to encounter in this chapter is talking about displacement rather than distance, and velocity rather than speed. For this reason, we have included a brief outline of the language of motion at the start of the chapter.

In Example 4 part **b**, we are given a velocity function, and asked to find the distance travelled in the first 4 seconds. In answering this question, we choose the initial displacement to be zero. We do this because an initial displacement is not given in the question, and the choice of initial displacement of the particle does not affect its distance travelled. Setting an initial displacement allows us to perform the distance calculations without involving a constant of integration c.

In Section D, a Discussion asks students to explain why the "sign test" for speed works, by considering various scenarios. The students should use the fact that the speed is the magnitude of the velocity. So, for example, if the velocity and acceleration are both negative, this means that the velocity is negative, and its value is decreasing, which means its *magnitude* is increasing, and so its speed is increasing.

CHAPTER 24: MACLAURIN SERIES

- **A** Maclaurin series
- **B** Convergence
- **C** Composite functions
- **D** Addition and subtraction
- **E** Differentiation and integration
- **F** Multiplication
- **G** Division

Syllabus reference: AHL 5.19

This chapter may be unfamiliar to teachers who did not teach the Calculus Option in the previous Mathematics HL course.

Compared to the treatment of the Calculus Option, we have tried to give a less formal presentation of the ideas behind Maclaurin series, so that students can better understand the ideas behind them, without getting overwhelmed with notation.

We start by considering the more general Taylor series, and then indicate that we are dealing with the special case where $a = 0$.

We have moved the treatment of the remainder term and the error to an online Activity at the end of Section A, as it does not appear to be part of the course. Our reading of the syllabus suggests that the focus is on performing operations with Maclaurin series, rather than assessing their accuracy.

That being said, we have included a section about the convergence of Maclaurin series. Convergence is not explicitly mentioned in the syllabus, but it seems unreasonable to present these series without some notion that they converge for some values of x and not for others. Doing so would be as incomplete as talking about the sum of an infinite geometric series without mentioning that the common ratio r must satisfy $-1 < r < 1$.

The Maclaurin series for basic functions are given in the formula booklet. However, we have included several questions asking students to prove these results, as we think this would be a perfectly valid exam question.

An important skill in this chapter is to use the Maclaurin series given in the formula booklet to find related Maclaurin series. For example, to find the Maclaurin series for $\sin(2x)$, we simply replace x by $2x$ in the Maclaurin series for $\sin x$.

It will be important to remind students of the motivation for Maclaurin series. Students may wonder why they would want to approximate the function, when they already have the exact function. It should be emphasised that expressing functions in terms of polynomial approximations is powerful, as operations such as addition, multiplication, differentiation, and integration are much easier to perform on polynomials.

CHAPTER 25: DIFFERENTIAL EQUATIONS

A Differential equations
B Euler's method for numerical integration
C Differential equations of the form $\frac{dy}{dx} = f(x)$
D Separable differential equations
E Logistic growth
F Homogeneous differential equations $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$
G The integrating factor method
H Maclaurin series developed from a differential equation

Syllabus reference: AHL 5.18, AHL 5.19

In Section A, it is important to establish the concept of differential equations, in particular that the solution to the differential equation is an *equation* such as $y = x^2 + x$, rather than a number such as $x = 5$. This understanding will set up students well for the remainder of the chapter. It may be useful to remind students of their integration work in Chapter 21, when they would, for example, find y if $\frac{dy}{dx} = 2x^3 - 4$. When they did this, they were effectively solving a differential equation.

We present Euler's method quite early in the chapter, so it does not interfere with the flow of solving differential equations analytically later in the chapter. In the Discussion at the end of Section B, students should find that the accuracy of the solution decreases as the number of iterations increases. This is because the error between the approximation and the exact solution is likely to build as we progress through the iterations. When considering what can go wrong with Euler's method, students should consider what happens if $\frac{dy}{dx}$ is undefined for a particular (x, y) pair used during the iteration.

In Section E, we again use partial fractions to write a rational function in an integrable form. This step is first shown in Example 13. It should be easy for students to verify this step, and it would be useful for them to work through the decomposition in full.

The final section provides a useful application of Maclaurin series. Instead of being given the exact function first, we are given its derivative, and by further differentiation we can obtain more terms of the Maclaurin polynomial expansion of y. This would allow us to obtain an accurate approximation for y, even if we could not solve the differential equation analytically.

CHAPTER 26: BIVARIATE STATISTICS

A Association between numerical variables
B Pearson's product-moment correlation coefficient
C Line of best fit by eye
D The least squares regression line
E The regression line of x against y

Syllabus reference: SL 4.4, SL 4.10

Most of this chapter is similar in style and presentation as in previous books and the MYP series.

In Section E (The least squares regression line), we delve deeper into the regression procedure and provide formulae for the regression coefficients. An online-only derivation of the regression coefficients is also provided for the students who are more algebraically inclined. Of course in exercise questions, students are encouraged to use technology to find the coefficients.

Section F (The regression line of x against y) is new. We motivate the need for the line of x against y in terms of accuracy/precision and make reference to the variable with which the error of the measurement is associated (and hence the *direction* that the distance from the regression line is measured).

CHAPTER 27: DISCRETE RANDOM VARIABLES

A Random variables
B Discrete probability distributions
C Expectation
D Variance and standard deviation
E Properties of $aX + b$
F The binomial distribution
G Using technology to find binomial probabilities
H The mean and standard deviation of a binomial distribution

Syllabus reference: SL 4.7, SL 4.8, AHL 4.14

We start the chapter with an introduction to the concept of (discrete) random variables and their probability distributions. If you are going through the Mathematics: Core Topics HL book and this book in chapter order, it will have been a long time since the students have seen probability. Before starting this chapter, it may be beneficial to briefly revise key probability concepts as they are assumed throughout this chapter and Chapter 28 (Continuous random variables).

Section C (Expectation) continues on directly from Section E (Making predictions with probability) from Chapter 11 of the Mathematics: Core Topics HL book.

Section D extends the ideas of variance and standard deviation, covered in Chapter 13, Section J of the Mathematics: Core Topics HL book, to random variables. Since we are considering variance/standard deviation of *random variables*, the aforementioned statistics are treated as *population* values.

In Section C.2, a Discussion asks students whether we would expect a gambling game to be "fair". Students should recognise that we would not expect gambling games to be "fair", otherwise the operator of the game would not make a profit. A useful direction to lead students would be to ask whether the word "fair" in the mathematical sense is equivalent to how the word is used in everyday life, and whether the fact that gambling games are not "fair" implies that the operators are being underhanded or deceptive.

Combinations and binomial coefficients are essential for the introduction of the binomial distribution and the formulation of its probability mass function. So ensuring that students are familiar with binomial coefficient notation is important.

CHAPTER 28: CONTINUOUS RANDOM VARIABLES

A Probability density functions
B Measures of centre and spread
C The normal distribution
D Calculating normal probabilities
E The standard normal distribution
F Normal quantiles

Syllabus reference: SL 4.9, SL 4.12, AHL 4.14

The chapter begins with an introduction to continuous random variables via probability density functions (PDFs). PDFs will very likely be quite conceptually difficult to understand, hence why we opt to introduce them via Investigation 1.

In this Investigation, we first motivate the PDF as the derivative of the cumulative distribution function (CDF) which is directly analogous to cumulative frequency curves studied in Chapter 13 of the Mathematics: Core Topics HL book. Once we have established this relationship, we turn it around and ask "how do we find probabilities given a probability density function?" Given the extensive study of Calculus in the prior chapters, the conclusion that "probability $=$ area under (PDF) curve" should not be too difficult to make.

To help clarify the difference between probability density functions and probability mass functions, the following may be useful:

- Probability density function $\to$ density $\to$ rate $\to$ for continuous random variables
- Probability mass function $\to$ mass $\to$ quantity $\to$ for discrete random variables

Section B (Measures of centre and spread) extends the ideas in Sections C, D, and E from Chapter 27 to continuous random variables.

Throughout Section B, we include questions relating to the exponential distribution which has no upper bound on its domain. Hence in calculating probabilities associated with it, we obtain improper integrals. See the notes for Chapter 22, Section I.

Section C introduces the normal distribution by focussing on how the normal distribution arises and exploring the *shape* of the distribution.

Probability calculations are treated separately in the following section. The standard normal distribution and quantiles sections have not seen much change from the previous Mathematics HL book.