Chapter 3

Algebraic expansion and simplification

Contents:

A Collecting like terms
B Product notation
C The distributive law
D The product \((a + b)(c + d)\)
E Difference of two squares
F Perfect squares expansion
G Further expansion
H The binomial expansion
The study of algebra is an important part of the problem solving process. When we convert real life problems into algebraic equations, we often obtain expressions that need to be expanded and simplified.

**OPENING PROBLEM**

Ethel is planning a rectangular flower bed with a lawn of constant width around it. The lawn’s outer boundary is also rectangular. The shorter side of the flower bed is $x$ m long.

- If the flower bed’s length is 4 m longer than its width, what is its width?
- If the width of the lawn’s outer boundary is double the width of the flower bed, what are the dimensions of the flower bed?
- Wooden strips form the boundaries of the flower bed and lawn. Find, in terms of $x$, the total length $L$ of wood required.

**COLLECTING LIKE TERMS**

In algebra, like terms are terms which contain the same variables (or letters) to the same indices.

For example:
- $xy$ and $-2xy$ are like terms.
- $x^2$ and $3x$ are unlike terms because the powers of $x$ are not the same.

Algebraic expressions can often be simplified by adding or subtracting like terms. We call this collecting like terms.

Consider $2a + 3a = \frac{a + a + a + a}{2 \text{ lots of } a} = \frac{5a}{3 \text{ lots of } a}$

**Example 1**

Where possible, simplify by collecting the terms:

<table>
<thead>
<tr>
<th></th>
<th>a $4x + 3x$</th>
<th>b $5y - 2y$</th>
<th>c $2a - 1 + a$</th>
<th>d $mn - 2mn$</th>
<th>e $a^2 - 4a$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$4x + 3x = 7x$</td>
<td>$5y - 2y = 3y$</td>
<td>$2a - 1 + a = 3a - 1$</td>
<td>$mn - 2mn = -mn$</td>
<td>$a^2 - 4a$ cannot be simplified since $a^2$ and $-4a$ are unlike terms.</td>
</tr>
</tbody>
</table>
Simplify by collecting like terms:

\[
\begin{align*}
\text{a} & \quad -a - 1 + 3a + 4 \\
& \quad = -a + 3a - 1 + 4 \\
& \quad = 2a + 3 \\
\text{b} & \quad 5a - b^2 + 2a - 3b^2 \\
& \quad = 5a + 2a - b^2 - 3b^2 \\
& \quad = 7a - 4b^2
\end{align*}
\]

\{ -a and 3a are like terms \\
-1 and 4 are like terms \}

\{ 5a and 2a are like terms \\
-b^2 and -3b^2 are like terms \}

**Example 2**

**EXERCISE 3A**

1 Simplify, where possible, by collecting like terms:

\[
\begin{align*}
\text{a} & \quad 5 + a + 4 \\
\text{b} & \quad 6 + 3 + a \\
\text{c} & \quad m - 2 + 5 \\
\text{d} & \quad x + 1 + x \\
\text{e} & \quad f + f - 3 \\
\text{f} & \quad 5a + a \\
\text{g} & \quad x^2 + 2x \\
\text{h} & \quad d^2 + d^2 + d \\
\text{i} & \quad 2a + 3a - 5 \\
\text{j} & \quad 2a + 3a - a \\
\text{k} & \quad 4xy + xy \\
\text{l} & \quad 3x^2z - x^2z \\
\end{align*}
\]

2 Simplify, where possible:

\[
\begin{align*}
\text{a} & \quad 7a - 7a \\
\text{b} & \quad 7a - a \\
\text{c} & \quad 7a - 7 \\
\text{d} & \quad xy + 2xy \\
\text{e} & \quad cd - 2cd \\
\text{f} & \quad 4p^2 - p^2 \\
\text{g} & \quad x + 3 + 2x + 4 \\
\text{h} & \quad 2 + a + 3a - 4 \\
\text{i} & \quad 2y - x + 3y + 3x \\
\text{j} & \quad 3m^2 + 2m - m^2 - m \\
\text{k} & \quad ab + 4 - 3 + 2ab \\
\text{l} & \quad x^2 + 2x - x^2 - 5 \\
\text{m} & \quad x^2 + 5x + 2x^2 - 3x \\
\text{n} & \quad ab + b + a + 4 \\
\text{o} & \quad 2x^2 - 3x - x^2 - 7x \\
\end{align*}
\]

3 Simplify, where possible:

\[
\begin{align*}
\text{a} & \quad 4x + 6 - x - 2 \\
\text{b} & \quad 2c + d - 2cd \\
\text{c} & \quad 3ab - 2ab + ba \\
\text{d} & \quad x^2 + 2x^2 + 2x^2 - 5 \\
\text{e} & \quad p^2 - 6 + 2p^2 - 1 \\
\text{f} & \quad 3a + 7 - 2a - 10 \\
\text{g} & \quad -3a + 2b - a - b \\
\text{h} & \quad a^2 + 2a - a^3 \\
\text{i} & \quad 2a^2 - a^3 - a^2 + 2a^3 \\
\text{j} & \quad 4xy - x - y \\
\text{k} & \quad xy^2 + x^2y + x^2y \\
\text{l} & \quad 4x^3 - 2x^2 - x^3 - x^2
\end{align*}
\]

**B**

**PRODUCT NOTATION**

In algebra we agree:

- to leave out the “×” signs between any multiplied quantities provided that at least one of them is an unknown (letter)
- to write numerals (numbers) first in any product
- where products contain two or more letters, we write them in alphabetical order.

For example:

- \(2a\) is used rather than \(2 \times a\) or \(a2\)
- \(2ab\) is used rather than \(2ba\).
ALGEBRAIC PRODUCTS

The product of two or more factors is the result obtained by multiplying them together.

Consider the factors \(-3x\) and \(2x^2\). Their product \(-3x \times 2x^2\) can be simplified by following the steps below:

Step 1: Find the product of the signs.
Step 2: Find the product of the numerals or numbers.
Step 3: Find the product of the variables or letters.

So, \(-3x \times 2x^2 = -6x^3\)

For \(-3x\), the sign is \(-\), the numeral is \(3\), and the variable is \(x\).

\[
\begin{align*}
\text{Step 1} & : \quad (-) \times (+) = (-) \\
\text{Step 2} & : \quad 3 \times 2 = 6 \\
\text{Step 3} & : \quad x \times x^2 = x^3 \\
\end{align*}
\]

Example 3

Simplify the following products:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>(-3 \times 4x)</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>(-12x)</td>
<td>b</td>
</tr>
</tbody>
</table>

EXERCISE 3B

1 Write the following algebraic products in simplest form:
   a | \(c \times b\)  | b | \(a \times 2 \times b\)  | c | \(y \times xy\)  | d | \(pq \times 2q\)  

2 Simplify the following:
   a | \(2 \times 3x\)  | b | \(4x \times 5\)  | c | \(-2 \times 7x\)  | d | \(3 \times -2x\)  
   e | \(2x \times x\)  | f | \(3x \times 2x\)  | g | \(-2x \times x\)  | h | \(-3x \times 4\)  
   i | \(-2x \times -x\)  | j | \(-3x \times x^2\)  | k | \(-x^2 \times -2x\)  | l | \(3d \times -2d\)  
   m | \((-a)^2\)  | n | \((-2a)^2\)  | o | \(2a^2 \times a^2\)  | p | \(a^2 \times -3a\)  

3 Simplify the following:
   a | \(2 \times 5x + 3x \times 4\)  | b | \(5 \times 3x - 2y \times y\)  | c | \(3 \times x^2 + 2x \times 4x\)  
   d | \(a \times 2b + b \times 3a\)  | e | \(4 \times x^2 - 3x \times x\)  | f | \(3x \times y - 2x \times 2y\)  
   g | \(3a \times b + 2a \times 2b\)  | h | \(4c \times d - 3c \times 2d\)  | i | \(3a \times b - 2c \times a\)  

For \(-3x\), the sign is \(-\), the numeral is \(3\), and the variable is \(x\).
Consider the expression $2(x + 3)$. We say that 2 is the coefficient of the expression in the brackets. We can expand the brackets using the distributive law:

$$a(b + c) = ab + ac$$

The distributive law says that we must multiply the coefficient by each term within the brackets, and add the results.

**Geometric Demonstration:**

The overall area is $a(b + c)$.

However, this could also be found by adding the areas of the two small rectangles, i.e., $ab + ac$.

So, $a(b + c) = ab + ac$  \{equating areas\}

---

**Example 4**

Expand the following:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$3(4x + 1)$</td>
<td>$2x(5 - 2x)$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>a</td>
<td>$3(4x + 1)$</td>
<td>$2x(5 - 2x)$</td>
</tr>
<tr>
<td>a</td>
<td>$= 3 \times 4x + 3 \times 1$</td>
<td>$= 2x(5 + -2x)$</td>
</tr>
<tr>
<td>a</td>
<td>$= 12x + 3$</td>
<td>$= 2x \times 5 + 2x \times -2x$</td>
</tr>
<tr>
<td>a</td>
<td>$= 10x - 4x^2$</td>
<td>$= -2x^2 + 6x$</td>
</tr>
</tbody>
</table>

With practice, we do not need to write all of these steps.

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**Example 5**

Expand and simplify:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$2(3x - 1) + 3(5 - x)$</td>
<td>$x(2x - 1) - 2x(5 - x)$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>a</td>
<td>$2(3x - 1) + 3(5 - x)$</td>
<td>$x(2x - 1) - 2x(5 - x)$</td>
</tr>
<tr>
<td>a</td>
<td>$= 6x - 2 + 15 - 3x$</td>
<td>$= 2x^2 - x - 10x + 2x^2$</td>
</tr>
<tr>
<td>a</td>
<td>$= 3x + 13$</td>
<td>$= 4x^2 - 11x$</td>
</tr>
</tbody>
</table>

Notice in b that the minus sign in front of $2x$ affects both terms inside the following bracket.
EXERCISE 3C

1 Expand and simplify:
   \[ a \] \( 3(x + 1) \)  \[ b \] \( 2(5 - x) \)  \[ c \] \(- (x + 2) \)  \[ d \] \(- (3 - x) \)
   \[ e \] \( 4(a + 2b) \)  \[ f \] \( 3(2x + y) \)  \[ g \] \( 5(x - y) \)  \[ h \] \( 6(-x^2 + y^2) \)
   \[ i \] \(- 2(x + 4) \)  \[ j \] \(- 3(2x - 1) \)  \[ k \] \( x(x + 3) \)  \[ l \] \( 2x(x - 5) \)
   \[ m \] \(- 3(x + 2) \)  \[ n \] \(- 4(x - 3) \)  \[ o \] \(- (3 - x) \)  \[ p \] \(- 2(x - y) \)
   \[ q \] \( a(a + b) \)  \[ r \] \(- a(a - b) \)  \[ s \] \( x(2x - 1) \)  \[ t \] \( 2x(x^2 - x - 2) \)

2 Expand and simplify:
   \[ a \] \( 1 + 2(x + 2) \)  \[ b \] \( 13 - 4(x + 3) \)  \[ c \] \( 3(x - 2) + 5 \)
   \[ d \] \( 4(3 - x) - 10 \)  \[ e \] \( x(x - 1) + x \)  \[ f \] \( 2x(3 - x) + x^2 \)
   \[ g \] \( 2a(b - a) + 3a^2 \)  \[ h \] \( 4x - 3x(x - 1) \)  \[ i \] \( 7x^2 - 5x(x + 2) \)

3 Expand and simplify:
   \[ a \] \( 3(x - 4) + 2(5 + x) \)  \[ b \] \( 2a + (a - 2b) \)  \[ c \] \( 2a - (a - 2b) \)
   \[ d \] \( 3(y + 1) + 6(2 - y) \)  \[ e \] \( 2(y - 3) - 4(2y + 1) \)  \[ f \] \( 3x - 4(2 - 3x) \)
   \[ g \] \( 2(b - a) + 3(a + b) \)  \[ h \] \( x(x + 4) + 2(x - 3) \)  \[ i \] \( x(x + 4) - 2(x - 3) \)
   \[ j \] \( x^2 + x(x - 1) \)  \[ k \] \(- x^2 - x(x - 2) \)  \[ l \] \( x(x + y) - y(x + y) \)
   \[ m \] \(- 4(x - 2) - (3 - x) \)  \[ n \] \( 5(2x - 1) - (2x + 3) \)  \[ o \] \( 4x(x - 3) - 2x(5 - x) \)

D THE PRODUCT \((a + b)(c + d)\)

Consider the product \((a + b)(c + d)\).

It has two factors, \((a + b)\) and \((c + d)\).

We can evaluate this product by using the distributive law several times.

\[
(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd
\]

So, \((a + b)(c + d) = ac + ad + bc + bd\)

The final result contains four terms:

- **ac** is the product of the **First terms of each bracket**.
- **ad** is the product of the **Outer terms of each bracket**.
- **bc** is the product of the **Inner terms of each bracket**.
- **bd** is the product of the **Last terms of each bracket**.

This is sometimes called the FOIL rule.
Expand and simplify: 
\[(x + 3)(x + 2)\]

\[(x + 3)(x + 2) = x \times x + x \times 2 + 3 \times x + 3 \times 2\]
\[= x^2 + 2x + 3x + 6\]
\[= x^2 + 5x + 6\]

Expand and simplify: 
\[(x + 3)(x - 3)\]

\[(x + 3)(x - 3) = x \times x + x \times (-3) + 3 \times x + 3 \times (-3)\]
\[= x^2 - 3x + 3x - 9\]
\[= x^2 - 9\]

Expand and simplify: 
\[(3x - 5)(3x + 5)\]

\[(3x - 5)(3x + 5) = 9x^2 + 15x - 15x - 25\]
\[= 9x^2 - 25\]

EXERCISE 3D

1. Consider the figure alongside:
   Give an expression for the area of:
   - rectangle 1
   - rectangle 2
   - rectangle 3
   - rectangle 4
   - the overall rectangle.
   What can you conclude?

2. Use the rule \((a + b)(c + d) = ac + ad + bc + bd\) to expand and simplify:
   \(a\) \((x + 3)(x + 7)\)
   \(b\) \((x + 5)(x - 4)\)
   \(c\) \((x - 3)(x + 6)\)
   \(d\) \((x + 2)(x - 2)\)
   \(e\) \((x - 8)(x + 3)\)
   \(f\) \((2x + 1)(3x + 4)\)
   \(g\) \((1 - 2x)(4x + 1)\)
   \(h\) \((4 - x)(2x + 3)\)
   \(i\) \((3x - 2)(1 + 2x)\)
   \(j\) \((5 - 3x)(5 + x)\)
   \(k\) \((7 - x)(4x + 1)\)
   \(l\) \((5x + 2)(5x + 2)\)

Example 8

Expand and simplify:
\(a\) \((x + 3)(x - 3)\)
\(b\) \((3x - 5)(3x + 5)\)

\(a\) \((x + 3)(x - 3) = x^2 - 3x + 3x - 9\)
\[= x^2 - 9\]

\(b\) \((3x - 5)(3x + 5) = 9x^2 + 15x - 15x - 25\)
\[= 9x^2 - 25\]
3 Expand and simplify:
   a \((x + 2)(x - 2)\)
   b \((a - 5)(a + 5)\)
   c \((4 + x)(4 - x)\)
   d \((2x + 1)(2x - 1)\)
   e \((5a + 3)(5a - 3)\)
   f \((4 + 3a)(4 - 3a)\)

Example 9

Expand and simplify:
   a \((3x + 1)^2\)
   b \((2x - 3)^2\)

   a \((3x + 1)^2\)
   = \((3x + 1)(3x + 1)\)
   = \(9x^2 + 3x + 3x + 1\)
   = \(9x^2 + 6x + 1\)

   b \((2x - 3)^2\)
   = \((2x - 3)(2x - 3)\)
   = \(4x^2 - 6x - 6x + 9\)
   = \(4x^2 - 12x + 9\)

4 Expand and simplify:
   a \((x + 3)^2\)
   b \((x - 2)^2\)
   c \((3x - 2)^2\)
   d \((1 - 3x)^2\)
   e \((3 - 4x)^2\)
   f \((5x - y)^2\)

E DIFFERENCE OF TWO SQUARES

\(a^2\) and \(b^2\) are perfect squares and so \(a^2 - b^2\) is called the difference of two squares.

Notice that \((a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2\)

the middle two terms add to zero

Thus, \((a + b)(a - b) = a^2 - b^2\)

Geometric Demonstration:
Consider the figure alongside:
The shaded area
\(= \) area of large square \(\) area of small square
\(= a^2 - b^2\)

Cutting along the dotted line and flipping (2) over,
we can form a rectangle.
The rectangle’s area is \((a + b)(a - b)\).

\(\therefore \ (a + b)(a - b) = a^2 - b^2\)
ALGEBRAIC EXPANSION AND SIMPLIFICATION (Chapter 3)

Example 10

Expand and simplify:

a. \((x + 5)(x - 5)\)

\[ = x^2 - 5^2 \]
\[ = x^2 - 25 \]

b. \((3 - y)(3 + y)\)

\[ = 3^2 - y^2 \]
\[ = 9 - y^2 \]

Example 11

Expand and simplify:

a. \((2x - 3)(2x + 3)\)

\[ = (2x)^2 - 3^2 \]
\[ = 4x^2 - 9 \]

b. \((5 - 3y)(5 + 3y)\)

\[ = 5^2 - (3y)^2 \]
\[ = 25 - 9y^2 \]

Example 12

Expand and simplify:

\((3x + 4y)(3x - 4y)\)

\[ = (3x)^2 - (4y)^2 \]
\[ = 9x^2 - 16y^2 \]

EXERCISE 3E

1. Expand and simplify using the rule \((a + b)(a - b) = a^2 - b^2:\)
   a. \((x + 2)(x - 2)\)
   b. \((x - 2)(x + 2)\)
   c. \((2 + x)(2 - x)\)
   d. \((2 - x)(2 + x)\)
   e. \((x + 1)(x - 1)\)
   f. \((1 - x)(1 + x)\)
   g. \((x + 7)(x - 7)\)
   h. \((c + 8)(c - 8)\)
   i. \((d - 5)(d + 5)\)
   j. \((x + y)(x - y)\)
   k. \((4 + d)(4 - d)\)
   l. \((5 + e)(5 - e)\)

2. Expand and simplify using the rule \((a + b)(a - b) = a^2 - b^2:\)
   a. \((2x - 1)(2x + 1)\)
   b. \((3x + 2)(3x - 2)\)
   c. \((4y - 5)(4y + 5)\)
   d. \((2y + 5)(2y - 5)\)
   e. \((3x + 1)(3x - 1)\)
   f. \((1 - 3x)(1 + 3x)\)
   g. \((2 - 5y)(2 + 5y)\)
   h. \((3 + 4a)(3 - 4a)\)
   i. \((4 + 3a)(4 - 3a)\)

3. Expand and simplify using the rule \((a + b)(a - b) = a^2 - b^2:\)
   a. \((2a + b)(2a - b)\)
   b. \((a - 2b)(a + 2b)\)
   c. \((4x + y)(4x - y)\)
   d. \((4x + 5y)(4x - 5y)\)
   e. \((2x + 3y)(2x - 3y)\)
   f. \((7x - 2y)(7x + 2y)\)
INVESTIGATION THE PRODUCT OF THREE CONSECUTIVE INTEGERS

Con was trying to multiply $19 \times 20 \times 21$ without a calculator. Aimee told him to ‘cube the middle integer and then subtract the middle integer’ to get the answer.

What to do:

1. Find $19 \times 20 \times 21$ using a calculator.
2. Find $20^3 - 20$ using a calculator. Does Aimee’s rule seem to work?
3. Check that Aimee’s rule works for the following products:
   a. $4 \times 5 \times 6$
   b. $9 \times 10 \times 11$
   c. $49 \times 50 \times 51$
4. Let the middle integer be $x$, so the other integers must be $(x - 1)$ and $(x + 1)$. Find the product $(x - 1) \times x \times (x + 1)$ by expanding and simplifying. Have you proved Aimee’s rule?

Hint: Use the difference between two squares expansion.

PERFECT SQUARES EXPANSION

$(a + b)^2$ and $(a - b)^2$ are called perfect squares.

Notice that 

\[
(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 \quad \text{using ‘FOIL’}
\]

Thus, we can state the perfect square expansion rule:

\[
(a + b)^2 = a^2 + 2ab + b^2
\]

We can remember the rule as follows:

Step 1: Square the first term.

Step 2: Add twice the product of the first and last terms.

Step 3: Add on the square of the last term.

Notice that 

\[
(a - b)^2 = (a + (-b))^2 = a^2 + 2a(-b) + (-b)^2 = a^2 - 2ab + b^2
\]

Once again, we have the square of the first term, twice the product of the first and last terms, and the square of the last term.
Expand and simplify:

**Example 13**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>a</strong></td>
<td>(x + 3)^2</td>
</tr>
<tr>
<td><strong>b</strong></td>
<td>(x - 5)^2</td>
</tr>
</tbody>
</table>

**Self Tutor**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| a | (x + 3)^2  
  = x^2 + 2 \times x \times 3 + 3^2  
  = x^2 + 6x + 9 |
| b | (x - 5)^2  
  = (x + -5)^2  
  = x^2 + 2 \times x \times (-5) + (-5)^2  
  = x^2 - 10x + 25 |

Expand and simplify using the perfect square expansion rule:

**Example 14**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a</strong></td>
<td>(5x + 1)^2</td>
</tr>
<tr>
<td><strong>b</strong></td>
<td>(4 - 3x)^2</td>
</tr>
</tbody>
</table>

**Self Tutor**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| a | (5x + 1)^2  
  = (5x)^2 + 2 \times 5x \times 1 + 1^2  
  = 25x^2 + 10x + 1 |
| b | (4 - 3x)^2  
  = (4 + -3x)^2  
  = 4^2 + 2 \times 4 \times (-3x) + (-3x)^2  
  = 16 - 24x + 9x^2 |

**EXERCISE 3F**

1. Consider the figure alongside:
   Give an expression for the area of:
   - a square 1  
   - b rectangle 2  
   - c rectangle 3  
   - d square 4  
   - e the overall square.

   What can you conclude?

2. Use the rule \((a+b)^2 = a^2 + 2ab + b^2\) to expand and simplify:
   - a \((x + 5)^2\)  
   - b \((x + 4)^2\)  
   - c \((x + 7)^2\)  
   - d \((a + 2)^2\)  
   - e \((3 + c)^2\)  
   - f \((5 + x)^2\)

3. Expand and simplify using the perfect square expansion rule:
   - a \((x - 3)^2\)  
   - b \((x - 2)^2\)  
   - c \((y - 8)^2\)  
   - d \((a - 7)^2\)  
   - e \((5 - x)^2\)  
   - f \((4 - y)^2\)

4. Expand and simplify using the perfect square expansion rule:
   - a \((3x + 4)^2\)  
   - b \((2a - 3)^2\)  
   - c \((3y + 1)^2\)  
   - d \((2x - 5)^2\)  
   - e \((3y - 5)^2\)  
   - f \((7 + 2a)^2\)  
   - g \((1 + 5x)^2\)  
   - h \((7 - 3y)^2\)  
   - i \((3 + 4a)^2\)
Example 15

Expand and simplify:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$(2x^2 + 3)^2$</td>
</tr>
<tr>
<td>b</td>
<td>$5 - (x + 2)^2$</td>
</tr>
</tbody>
</table>

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$(2x^2 + 3)^2$</td>
</tr>
<tr>
<td></td>
<td>$= (2x^2)^2 + 2 \times 2x^2 \times 3 + 3^2$</td>
</tr>
<tr>
<td></td>
<td>$= 4x^4 + 12x^2 + 9$</td>
</tr>
<tr>
<td>b</td>
<td>$5 - (x + 2)^2$</td>
</tr>
<tr>
<td></td>
<td>$= 5 - [x^2 + 4x + 4]$</td>
</tr>
<tr>
<td></td>
<td>$= 5 - x^2 - 4x - 4$</td>
</tr>
<tr>
<td></td>
<td>$= 1 - x^2 - 4x$</td>
</tr>
</tbody>
</table>

5. Expand and simplify:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$(x^2 + 2)^2$</td>
</tr>
<tr>
<td>b</td>
<td>$(y^2 - 3)^2$</td>
</tr>
<tr>
<td>c</td>
<td>$(3a^2 + 4)^2$</td>
</tr>
<tr>
<td>d</td>
<td>$(1 - 2x^2)^2$</td>
</tr>
<tr>
<td>e</td>
<td>$(x^2 + y^2)^2$</td>
</tr>
<tr>
<td>f</td>
<td>$(x^2 - a^2)^2$</td>
</tr>
</tbody>
</table>

6. Expand and simplify:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$3x + 1 - (x + 3)^2$</td>
</tr>
<tr>
<td>b</td>
<td>$5x - 2 + (x - 2)^2$</td>
</tr>
<tr>
<td>c</td>
<td>$(x + 2)(x - 2) + (x + 3)^2$</td>
</tr>
<tr>
<td>d</td>
<td>$(x + 2)(x - 2) - (x + 3)^2$</td>
</tr>
<tr>
<td>e</td>
<td>$(3 - 2x)^2 - (x - 1)(x + 2)$</td>
</tr>
<tr>
<td>f</td>
<td>$(1 - 3x)^2 + (x + 2)(x - 3)$</td>
</tr>
<tr>
<td>g</td>
<td>$(2x + 3)(2x - 3) - (x + 1)^2$</td>
</tr>
<tr>
<td>h</td>
<td>$(4x + 3)(x - 2) - (2 - x)^2$</td>
</tr>
<tr>
<td>i</td>
<td>$(1 - x)^2 + (x + 2)^2$</td>
</tr>
<tr>
<td>j</td>
<td>$(1 - x)^2 - (x + 2)^2$</td>
</tr>
</tbody>
</table>

G. Further Expansion

In this section we expand more complicated expressions by repeated use of the expansion laws.

Consider the expansion of $(a + b)(c + d + e)$.

Now $(a + b)(c + d + e)$

$= (a + b)c + (a + b)d + (a + b)e$

$= ac + bc + ad + bd + ae + be$

Compare: $\square(c + d + e)$

Notice that there are 6 terms in this expansion and that each term within the first bracket is multiplied by each term in the second.

2 terms in the first bracket $\times$ 3 terms in the second bracket $\longrightarrow$ 6 terms in the expansion.

Example 16

Expand and simplify: $(2x + 3)(x^2 + 4x + 5)$

$= 2x^3 + 8x^2 + 10x$ \{all terms of 2nd bracket $\times$ 2x\}

$+ 3x^2 + 12x + 15$ \{all terms of 2nd bracket $\times$ 3\}

$= 2x^3 + 11x^2 + 22x + 15$ \{collecting like terms\}
Example 17

Expand and simplify: \((x + 2)^3\)

\[
(x + 2)^3 = (x + 2)(x + 2)^2
= (x + 2)(x^2 + 4x + 4)
= x^3 + 4x^2 + 4x + 2x^2 + 8x + 8
= x^3 + 6x^2 + 12x + 8
\]

Example 18

Expand and simplify:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>a</strong></td>
<td>(x(x + 1)(x + 2))</td>
</tr>
<tr>
<td><strong>b</strong></td>
<td>((x + 1)(x - 2)(x + 2))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>a</strong></th>
<th>(x(x + 1)(x + 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(= (x^2 + x)(x + 2))</td>
<td>{all terms in first bracket (\times x})</td>
</tr>
<tr>
<td>(= x^3 + 2x^2 + x^2 + 2x)</td>
<td>{expanding remaining factors}</td>
</tr>
<tr>
<td>(= x^3 + 3x^2 + 2x)</td>
<td>{collecting like terms}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>b</strong></th>
<th>((x + 1)(x - 2)(x + 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(= (x + 1)(x^2 - 4))</td>
<td>{difference of two squares}</td>
</tr>
<tr>
<td>(= x^3 - 4x + x^2 - 4)</td>
<td>{expanding factors}</td>
</tr>
</tbody>
</table>

EXERCISE 3G

1 Expand and simplify:

<table>
<thead>
<tr>
<th><strong>a</strong></th>
<th>((x + 3)(x^2 + x + 2))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>b</strong></td>
<td>((x + 4)(x^2 + x - 2))</td>
</tr>
<tr>
<td><strong>c</strong></td>
<td>((x + 2)(x^2 + x + 1))</td>
</tr>
<tr>
<td><strong>d</strong></td>
<td>((x + 5)(x^2 - x - 1))</td>
</tr>
<tr>
<td><strong>e</strong></td>
<td>((2x + 1)(x^2 + x + 4))</td>
</tr>
<tr>
<td><strong>f</strong></td>
<td>((3x - 2)(x^2 - x - 3))</td>
</tr>
<tr>
<td><strong>g</strong></td>
<td>((x + 2)(2x^2 - x + 2))</td>
</tr>
<tr>
<td><strong>h</strong></td>
<td>((2x - 1)(3x^2 - x + 2))</td>
</tr>
</tbody>
</table>

2 Expand and simplify:

<table>
<thead>
<tr>
<th><strong>a</strong></th>
<th>((x + 1)^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>b</strong></td>
<td>((x + 3)^3)</td>
</tr>
<tr>
<td><strong>c</strong></td>
<td>((x - 1)^3)</td>
</tr>
<tr>
<td><strong>d</strong></td>
<td>((x - 3)^3)</td>
</tr>
<tr>
<td><strong>e</strong></td>
<td>((2x + 1)^3)</td>
</tr>
<tr>
<td><strong>f</strong></td>
<td>((3x - 2)^3)</td>
</tr>
</tbody>
</table>

3 Expand and simplify:

<table>
<thead>
<tr>
<th><strong>a</strong></th>
<th>(x(x + 2)(x + 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>b</strong></td>
<td>(x(x - 4)(x + 1))</td>
</tr>
<tr>
<td><strong>c</strong></td>
<td>(x(x - 3)(x - 2))</td>
</tr>
<tr>
<td><strong>d</strong></td>
<td>(2x(x + 3)(x + 1))</td>
</tr>
<tr>
<td><strong>e</strong></td>
<td>(2x(x - 4)(1 - x))</td>
</tr>
<tr>
<td><strong>f</strong></td>
<td>(-x(3 + x)(2 - x))</td>
</tr>
<tr>
<td><strong>g</strong></td>
<td>(-3x(2x - 1)(x + 2))</td>
</tr>
<tr>
<td><strong>h</strong></td>
<td>(x(1 - 3x)(2x + 1))</td>
</tr>
<tr>
<td><strong>i</strong></td>
<td>(2x^2(x - 1)^2)</td>
</tr>
</tbody>
</table>
Expand and simplify:

a \((x + 3)(x + 2)(x + 1)\)

b \((x - 2)(x - 1)(x + 4)\)

c \((x - 4)(x - 1)(x - 3)\)

d \((2x - 1)(x + 2)(x - 1)\)

e \((3x + 2)(x + 1)(x + 3)\)

f \((2x + 1)(2x - 1)(x + 4)\)

g \((1 - x)(3x + 2)(x - 2)\)

h \((x - 3)(1 - x)(3x + 2)\)

THE BINOMIAL EXPANSION

Consider \((a + b)^n\). We note that:

- \(a + b\) is called a binomial as it contains two terms.
- any expression of the form \((a + b)^n\) is called a power of a binomial.
- the binomial expansion of \((a + b)^n\) is obtained by writing the expression without brackets.

Now \((a + b)^3 = (a + b)^2(a + b)\):

\[
(a + b)^2 = (a^2 + 2ab + b^2)(a + b) = a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3
\]

So, the binomial expansion of \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\).

Example 19

Expand and simplify using the rule \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\).

a \((x + 2)^3\)

b \((2x - 1)^3\)

a We substitute \(a = x\) and \(b = 2\):

\[
(x + 2)^3 = x^3 + 3 \times x^2 \times 2 + 3 \times x \times 2^2 + 2^3 = x^3 + 6x^2 + 12x + 8
\]

b We substitute \(a = (2x)\) and \(b = (-1)\):

\[
(2x - 1)^3 = (2x)^3 + 3 \times (2x)^2 \times (-1) + 3 \times (2x) \times (-1)^2 + (-1)^3 = 8x^3 - 12x^2 + 6x - 1
\]

EXERCISE 3H

1 Use the binomial expansion for \((a + b)^3\) to expand and simplify:

a \((x + 1)^3\)

b \((a + 3)^3\)

c \((x + 5)^3\)

d \((x - 1)^3\)

e \((x - 2)^3\)

f \((x - 3)^3\)

g \((3 + a)^3\)

h \((3x + 2)^3\)

i \((2x + 3y)^3\)
2 Copy and complete the argument \((a + b)^4 = (a + b)(a + b)^3 = (a + b)(a^3 + 3a^2b + 3ab^2 + b^3)\) to expand and simplify:

3 Use the binomial expansion \((a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\) to expand and simplify:
   - a. \((x + 1)^4\)
   - b. \((y + 2)^4\)
   - c. \((3 + a)^4\)
   - d. \((b + 4)^4\)
   - e. \((x - 1)^4\)
   - f. \((y - 2)^4\)
   - g. \((3 - a)^4\)
   - h. \((b - 4)^4\)

4 Find the binomial expansion of \((a + b)^5\) by considering \((a + b)(a + b)^4\). Hence, write down the binomial expansion for \((a - b)^5\).

---

**REVIEW SET 3A**

1 Expand and simplify:
   - a. \(4x \times -8\)
   - b. \(5x \times 2x^2\)
   - c. \(-4x \times -6x\)
   - d. \(3x \times x - 2x^2\)
   - e. \(4a \times c + 3c \times a\)
   - f. \(2x^2 \times x - 3x \times x^2\)

2 Expand and simplify:
   - a. \(-3(x + 6)\)
   - b. \(2x(x^2 - 4)\)
   - c. \(2(x - 5) + 3(2 - x)\)
   - d. \(3(1 - 2x) - (x - 4)\)
   - e. \(2x - 3x(x - 2)\)
   - f. \(x(2x + 1) - 2x(1 - x)\)
   - g. \(x^2(x + 1) - x(1 - x^2)\)
   - h. \(9(a + b) - a(4 - b)\)

3 Expand and simplify:
   - a. \((3x + 2)(x - 2)\)
   - b. \((2x - 1)^2\)
   - c. \((4x + 1)(4x - 1)\)
   - d. \((5 - x)^2\)
   - e. \((3x - 7)(2x - 5)\)
   - f. \((x + 2)(x - 2)\)
   - g. \((3x + 5)^2\)
   - h. \(-(x - 2)^2\)
   - i. \(-2x(x - 1)^2\)

4 Expand and simplify:
   - a. \(5 + 2x - (x + 3)^2\)
   - b. \((x + 2)^3\)
   - c. \((3x - 2)(x^2 + 2x + 7)\)
   - d. \((x - 1)(x - 2)(x - 3)\)
   - e. \(x(x + 1)^3\)
   - f. \((x^2 + 1)(x - 1)(x + 1)\)

5 Explain how to use the given figure to show that \((a + b)^2 = a^2 + 2ab + b^2\).
REVIEW SET 3B

1 Expand and simplify:
   a $3x \times -2x^2$  
   b $2x^2 \times -3x$  
   c $-5x \times -8x$  
   d $(2x)^2$  
   e $(-3x^2)^2$  
   f $4x \times -x^2$

2 Expand and simplify:
   a $-7(2x - 5)$  
   b $2(x - 3) + 3(2 - x)$  
   c $-x(3 - 4x) - 2x(x + 1)$  
   d $2(3x + 1) - 5(1 - 2x)$  
   e $3x(x^2 + 1) - 2x^2(3 - x)$  
   f $3(2a + b) - 5(b - 2a)$

3 Expand and simplify:
   a $(2x + 5)(x - 3)$  
   b $(3x - 2)^2$  
   c $(2x + 3)(2x - 3)$  
   d $(x + 3)^3$  
   e $(2x - 3)^2$  
   f $(1 - 5x)(1 + 5x)$  
   g $(5 - 2x)^2$  
   h $-(x + 2)^2$  
   i $-3x(1 - x)^2$

4 Expand and simplify:
   a $(2x + 1)^2 - (x - 2)(3 - x)$  
   b $(x^2 - 4x + 3)(2x - 1)$  
   c $(x + 3)^3$  
   d $(x + 1)(x - 2)(x + 5)$  
   e $2x(x - 1)^3$  
   f $(4 - x^2)(x + 2)(x - 2)$

5 Use the binomial expansion $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:
   a $(2x + 1)^4$  
   b $(x - 3)^4$

6 What algebraic fact can you derive by considering the area of the given figure in two different ways?