Networks and matrices

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Review set 7
Networks are used to show how things are connected. They can be used to help solve problems in train scheduling, traffic flow, bed usage in hospitals, and project management.

The construction of networks belongs to the branch of mathematics known as Topology.

Various procedures known as algorithms are applied to the networks to find maximum or minimum solutions. This further study of the constructed networks belongs to the branch of mathematics known as Operations Research.

HISTORICAL NOTE

The theory of networks was developed centuries ago, but the application of the theoretical ideas is relatively recent.

Real progress was made during and after the Second World War as mathematicians, industrial technicians, and members of the armed services worked together to improve military operations.

Since then the range of applications has extended, from delivering mail to a neighbourhood to landing man on the moon.

Businesses and governments are increasingly using networks as a planning tool. They usually wish to find the optimum solution to a practical problem. This may be the solution which costs the least, uses the least time, or requires the smallest amount of material.

OPENING PROBLEMS

There are several common types of networking problem that we will be investigating in this chapter. For some types there are well-known algorithms that provide quick solutions. A few problems, however, remain a great challenge for 21st century mathematicians.

1 Shortest Path Problem

Dien and Phan are good friends who like to visit each other. The network alongside shows the roads between their houses. The times taken to walk each road are shown in minutes.

a Assuming there is no back-tracking, how many different paths are there from Dien’s house to Phan’s house?

b Which is the quickest route?

2 Shortest Connection Problem

A computer network relies on every computer being connected to every other. The connections do not have to be direct, i.e., a connection via another computer is fine. What is the minimum length of cable required to connect all computers to the network? The solid line on the network shows one possible solution, but not the shortest one.
3 Chinese Postman Problem
A postman needs to walk down every street in his district in order to deliver the mail. The numbers indicate the distances along the roads in hundreds of metres. The postman starts at the post office P, and returns there after delivering the mail. What route should the postman take to minimise the distance travelled?

4 Travelling Salesman Problem
A salesman leaves his home H in the morning, and over the course of his day he needs to attend meetings in towns A, B, C and D. At the end of the day he returns home. In which order should he schedule his meetings so that the distance he must travel is minimised?

In addition to the problems posed above, we will investigate how networks can be used to model projects with many steps. We will see how networks are related to matrices and learn how operations with matrices can be performed.

A network diagram or finite graph is a structure or model where things of interest are linked by physical connections or relationships.

The things of interest are represented by dots or circles and the connections or relationships represented by connecting lines.

For example, in the Travelling Salesman Problem (Opening Problem question 4) the dots represent the salesman’s house and the four towns. The connecting lines represent the roads between them.

TERMINOLOGY

- A graph or network is a set of points, some or all of which are connected by a set of lines.
- The points are known as nodes or vertices (singular vertex).
- In a graph, we can move from node to node along the lines. If we are allowed to move in either direction along the lines, the lines are called edges and the graph is undirected. Pairs of nodes that are directly connected by edges are said to be adjacent.

If we are only allowed to move in one direction along the lines, the lines are called arcs.

The graph is then known as a digraph or directed graph.
An edge or arc may sometimes be assigned a number called its **weight**. This may represent the cost, time, or distance required to travel along that line.

Note that the lengths of the arcs and edges are not drawn to scale, i.e., they are not in proportion to their weight.

The **degree** or **valence** of a vertex is the number of edges or arcs which touch it.

A **loop** starts and ends at the same vertex. It counts as one edge, but it contributes **two** to the degree of a vertex.

An **isolated** vertex has degree zero.

A **connected** graph is a graph in which it is possible to travel from every vertex to every other vertex by following edges. The graph is otherwise said to be **disconnected**.

A **complete** graph is a graph in which every vertex is adjacent to every other vertex.

A **simple** graph is a graph in which no vertex connects to itself, and every pair of vertices is directly connected by at most one edge.

A **path** is a connected sequence of edges or arcs showing a route that starts at one vertex and ends at another.

For example: A - B - E - D

A **circuit** is a path that starts and ends at the same vertex.

For example: A - B - C - E - A
An **Eulerian circuit** is a circuit that traverses each edge (or arc) exactly once.  
For example:  
A - B - C - B - E - C - D - E - A

A **Hamiltonian path** is a path which starts at one vertex and visits each vertex once and only once. It may not finish where it started.  
For example:  
A - E - B - C - D

A **Hamiltonian circuit** is a circuit which starts at one vertex, visits each vertex exactly once, and returns to where it started.  
For example:  
A - B - C - D - E

A **spanning tree** is a simple graph with no circuits which connects all vertices.  
For example:

![spanning tree](image)

is a spanning tree.

**EXERCISE 7A.1**

1 For the networks shown, state:  
   i the number of vertices  
   ii the number of edges  
   iii the degree of vertex B  
   iv whether the graph is simple or non-simple  
   v whether the graph is complete or not complete  
   vi whether the graph is connected or not connected.

![networks](image)

2 For the graph shown:  
   a name a path starting at A and ending at E  
   that does not “visit” any vertex more than once  
   b name a Hamiltonian path starting at A and finishing at E  
   c name a Hamiltonian circuit from A  
   d name a Hamiltonian circuit from D  
   e draw a spanning tree which includes the edges AB and AF.
3 For the network shown, draw as many spanning trees as you can find.

4 For the graph shown, list all possible Hamiltonian circuits starting at A.

5 A gas pipeline is to be constructed to link several towns in the country. Assuming the pipeline construction costs are the same everywhere in the region, the cheapest network is:
   
   A a Hamiltonian circuit
   B an Eulerian circuit
   C a spanning tree
   D a complete graph

6 Give real life examples of situations where a person may be interested in using
   
   a a Hamiltonian path
   b a Hamiltonian circuit

**INVESTIGATION 1**

**LEONHARD EULER AND THE BRIDGES OF KÖNIGSBERG**

The 18th century Swiss mathematician Leonhard Euler is famous in many areas of mathematics and science, including calculus, mechanics, optics, and astronomy.

In fact, Euler published more research papers than any other mathematician in history. However, nowhere was Euler’s contribution more significant than in topology. You have already met Eulerian circuits that were named in his honour. In this investigation we will meet another of his most famous contributions which is closely related to Eulerian circuits.

The town of Königsberg or Kaliningrad as it is now known, was situated on the river Pregel in Prussia. It had seven bridges linking two islands and the north and south banks of the river.

**What to do:**

1 The question is: could a tour be made of the town, returning to the original point, that crosses all of the bridges exactly once? A simplified map of Kaliningrad is shown alongside. Euler answered this question - can you?

2 Apparently, such a circuit is not possible. However, it would be possible if either one bridge was removed or one was added. Which bridge would you remove? Where on the diagram would you add a bridge?
3 For each of the following graphs determine:
   i the number of vertices with even degree
   ii the number of vertices with odd degree
   iii whether or not the graph has an Eulerian circuit.
      (Try to find an Eulerian circuit by trial and error.)

Summarise your results in a table:

<table>
<thead>
<tr>
<th>Network</th>
<th>No of vertices with even degree</th>
<th>No of vertices with odd degree</th>
<th>Eulerian circuit (Yes/No)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>e</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>f</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>g</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>h</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>i</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4 Can you see a simple rule to determine whether or not a graph has an Eulerian circuit? State the rule. Design your own networks or graphs to confirm the rule.

From **Investigation 1** you should have determined that:

An undirected network has an Eulerian circuit if and only if there are no vertices of odd degree.
**EXAMINING NETWORK DIAGRAMS**

Networks can be used to display a wide variety of information. They can be used as:
- flow charts showing the flow of fluid, traffic, or people
- precedence diagrams showing the order in which jobs must occur to complete a complex task
- maps showing the distances or times between locations
- family trees showing how people are related
- displays for sports results and many more things besides.

**EXERCISE 7A.2**

1. Consider the network showing which people are friends in a table tennis club.

   ![Table Tennis Network Diagram]

   a. What do the nodes of this graph represent?
   b. What do the edges represent?
   c. Who is Michael friendly with?
   d. Who has the most friends?
   e. How could we indicate that while Alan really likes Paula, Paula thinks Alan is a bit of a creep?

2. Alex is planning to water the garden with an underground watering system which has pop-up sprinklers. A network is to be drawn to show where the sprinklers are to be located.

   a. Could the network be considered to be directed?
   b. What are the nodes of the diagram?
   c. What are the edges?
   d. What problem might Alex be interested in solving?

3. At a school sports day, a round-robin ‘tug of war’ competition is held between the houses.

   a. What are the nodes of this graph?
   b. What do the edges represent?
   c. What is the significance of the degree of each node?
   d. Why, in this context, must we have a *complete* graph?
   e. How many matches must be played in total?
Using \( B \equiv \text{Blue}, R \equiv \text{Red}, \text{etc.} \) BR is the match between the Blues and the Reds. List all possible games using this method.

How could you indicate, on the network, who won each match?

Jon constructed a network model of his morning activities from waking up to leaving for school.

Write a brief account of Jon’s morning activities indicating the order in which events occur.

A roughly drawn map of the Riverland of SA is shown alongside. The distances between towns in kilometres are shown in black. The times in minutes required to travel between the towns are shown in red.

What are the nodes of the network?

What do we call the red and black numbers?

What is the shortest distance by road from Waikerie to Renmark according to the map?

Is it further to travel from Renmark to Kingston or Renmark to Loxton?

Is it quicker to travel from Renmark to Kingston or Renmark to Loxton?

The network alongside describes the distribution of water in a home. If we added weights to this network, describe three things they could represent.

Discuss the following:

- What types of computer network could be used in an office or school?
- What type of network would be suitable for a telephone system?
- What would the vertices and edges represent in each case?
Often we are given data in the form of a table, a list of jobs, or a verbal description. We need to use this information to construct a network that accurately displays the situation.

Example 1

Model a Local Area Network (LAN) of three computers connected through a server to a printer and scanner.

Example 2

Model the access to rooms on the first floor of this two-storey house as a network.

Note: The rooms are the nodes, the doorways are the edges.

**EXERCISE 7B.1**

1. Draw a network diagram to represent the roads between towns P, Q, R, S and T if:
   - Town P is 23 km from town Q and 17 km from town T
   - Town Q is 20 km from town R and 38 km from town S
   - Town R is 31 km from town S.

2. Draw a network diagram of friendships if:
   - A has friends B, D and F;
   - B has friends A, C and E;
   - C has friends D, E and F.
3 Draw a network diagram to represent how Australia’s states and territories share their borders. Represent each state as a node and each border as an edge.

**Australia’s states and territories**

4 a Model the room access on the first floor of the house plan below as a network diagram.

   - kitchen
   - dining
   - bedroom 3
   - hall
   - ensuite
   - family
   - bedroom 1
   - bedroom 2
   - bathroom

   **plan: first floor**

b Model access to rooms and outside for the ground floor of the house plan below as a network diagram. Consider outside as an external room represented by a single node.

   - entry
   - living
   - garage
   - roller door
   - study
   - bathroom
   - rumpus
   - laundry

   **plan: ground floor**
INVESTIGATION 2  

TOPOLOGICAL EQUIVALENCE

My house has ducted air conditioning in all of the main rooms. There are three ducts that run out from the main unit. One duct goes to the lounge room and on to the family room. The second duct goes to the study and on to the kitchen. The third duct goes upstairs, where it divides into three sub-ducts that service the three bedrooms.

What to do:
1. Draw a network to represent the ducting in my house.
2. Do you think your network provides a good idea of the layout of my house?
3. Compare your network to those drawn by your classmates. Check each network to make sure it correctly represents the description of my ducting. Do all of the correct networks look the same?
4. For a given networking situation, is there always a unique way to draw the network?

In Investigation 2 you should have found that there can be many ways to represent the same information.

Networks that look different but represent the same information are said to be **topologically equivalent** or **isomorphic**.

For example, the following networks are topologically equivalent:

Check to make sure each node has the correct connections.

EXERCISE 7B.2

1. Which of the networks in the diagrams following are topologically equivalent?
2 Label corresponding vertices of the following networks to show their topological equivalence:

![Network Diagram]

3 Draw a network diagram with the following specifications:
   - there are five nodes (or vertices)
   - two of the nodes each have two edges
   - one node has three edges and one node has four.

Remember when comparing your solution with others that networks may appear different but be topologically equivalent.

**PRECEDENCE NETWORKS**

Networks may be used to represent the steps involved in a project.

The building of a house, the construction of a newsletter, and cooking an evening meal all require many separate tasks to be completed.

Some of the tasks may happen *concurrently* (at the same time) while others are dependent upon the completion of another task.

If task B cannot begin until task A is completed, then task A is a **prerequisite** for task B.

For example, putting water in the kettle is a **prerequisite** to boiling it.

If we are given a list of tasks necessary to complete a project, we need to
   - write the tasks in the order in which they must be performed
   - determine any prerequisite tasks
   - construct a network to accurately represent the project.

Consider the tasks involved in making a cup of tea. They are listed below, along with their respective times (in seconds) for completion. A table like this is called a **precedence table** or an *activity table*.

<table>
<thead>
<tr>
<th></th>
<th>Description</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Retrieve the cups.</td>
<td>15</td>
</tr>
<tr>
<td>B</td>
<td>Place tea bags into cups.</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>Fill the kettle with water.</td>
<td>20</td>
</tr>
<tr>
<td>D</td>
<td>Boil the water in the kettle.</td>
<td>100</td>
</tr>
<tr>
<td>E</td>
<td>Pour the boiling water into the cups.</td>
<td>10</td>
</tr>
<tr>
<td>F</td>
<td>Add the milk and sugar.</td>
<td>15</td>
</tr>
<tr>
<td>G</td>
<td>Stir the tea.</td>
<td>10</td>
</tr>
</tbody>
</table>
Example 3

The steps involved in preparing a home-made pizza are listed below.

- A: Defrost the pizza base.
- B: Prepare the toppings.
- C: Place the sauce and toppings on the pizza.
- D: Heat the oven.
- E: Cook the pizza.

a Which tasks can be performed concurrently?
b Which tasks are prerequisite tasks?
c Draw a precedence table for the project.
d Draw a network diagram for the project.

a Tasks A, B and D may be performed concurrently, i.e., the pizza base could be defrosting and the oven could be heating up while the toppings are prepared.
b The toppings cannot be placed on the pizza until after the toppings have been prepared.

∴ task B is a prerequisite for task C.
The pizza cannot be cooked until everything else is done.
∴ tasks A, B, C and D are all prerequisites for task E.
c A precedence table shows the tasks and any prerequisite tasks.

d The network diagram may now be drawn.

EXERCISE 7B.3

1 The tasks for the following projects are not in the correct order. For each project write the correct order in which the tasks should be completed.

a Preparing an evening meal:
1 find a recipe 2 clean up 3 prepare ingredients
4 cook casserole 5 set table 6 serve meals

b Planting a garden:
1 dig the holes 2 purchase the trees 3 water the trees
4 plant the trees 5 decide on the trees required

2 Which tasks in question 1 could be performed concurrently?
3. The activities involved in preparing, barbecuing and serving a meal of hamburgers are given in the table alongside.

<table>
<thead>
<tr>
<th>Task</th>
<th>Prerequisite</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Gather ingredients</td>
<td></td>
</tr>
<tr>
<td>B Pre-heat barbecue</td>
<td></td>
</tr>
<tr>
<td>C Mix and shape hamburgers</td>
<td>A</td>
</tr>
<tr>
<td>D Cook hamburgers</td>
<td>B, C</td>
</tr>
<tr>
<td>E Prepare salad and rolls</td>
<td>A</td>
</tr>
<tr>
<td>F Assemble hamburgers and serve</td>
<td>D, E</td>
</tr>
</tbody>
</table>

Draw a network diagram to represent this project.

4. Your friend’s birthday is approaching and you decide to bake a cake. The individual tasks are listed below:

<table>
<thead>
<tr>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Mix the ingredients.</td>
</tr>
<tr>
<td>B Heat the oven.</td>
</tr>
<tr>
<td>C Place the cake mixture into the cake tin.</td>
</tr>
<tr>
<td>D Bake the cake.</td>
</tr>
<tr>
<td>E Cool the cake.</td>
</tr>
<tr>
<td>F Mix the icing.</td>
</tr>
<tr>
<td>G Ice the cake.</td>
</tr>
</tbody>
</table>

**a** Draw a precedence table for the project.

**b** Which tasks may be performed concurrently?

**c** Draw a network diagram for the project.

5. The construction of a back yard shed includes the following steps:

<table>
<thead>
<tr>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Prepare the area for the shed.</td>
</tr>
<tr>
<td>B Prepare the formwork for the concrete.</td>
</tr>
<tr>
<td>C Lay the concrete.</td>
</tr>
<tr>
<td>D Let concrete dry.</td>
</tr>
<tr>
<td>E Purchase the timber and iron sheeting.</td>
</tr>
<tr>
<td>F Build the timber frame.</td>
</tr>
<tr>
<td>G Fix iron sheeting to frame.</td>
</tr>
<tr>
<td>H Add window, door and flashing.</td>
</tr>
</tbody>
</table>

**a** Draw a precedence table for the project.

**b** Which tasks may be performed concurrently?

**c** Draw a network diagram for the project.

6. The separate steps involved in hosting a party are listed below:

<table>
<thead>
<tr>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Decide on a date to hold the party.</td>
</tr>
<tr>
<td>B Prepare invitations.</td>
</tr>
<tr>
<td>C Post the invitations.</td>
</tr>
<tr>
<td>D Wait for RSVPs.</td>
</tr>
<tr>
<td>E Clean and tidy the house.</td>
</tr>
<tr>
<td>F Organise food and drinks.</td>
</tr>
<tr>
<td>G Organise entertainment.</td>
</tr>
</tbody>
</table>
Draw a network diagram to model the project, indicating which tasks may be performed concurrently and any prerequisites that exist.

7 The tasks for cooking a breakfast of bacon and eggs, tea and toast are listed below:
- boil kettle; toast bread; collect plate and utensils; serve breakfast; gather ingredients;
- cook bacon and eggs; make tea; gather cooking utensils.

a Rearrange the tasks so that they are in the order that they should be completed.
b Which tasks can be performed concurrently?
c Construct a precedence table and network diagram for the project.

**DISCUSSION**

We have already seen how some tasks may be performed concurrently. What factors may determine whether the tasks are performed concurrently in practice? How might the number of available workers have an effect?

**PROBLEM SOLVING WITH NETWORKS**

In this section we will investigate a number of network problems, including:
- the number of paths through a network
- the shortest and longest paths through a network
- the shortest connection problem
- the Chinese Postman problem
- the Travelling Salesman Problem.

**NUMBER OF PATHS PROBLEMS**

It is often useful to know how many paths there are from one point on a network to another point. Such problems are usually associated with directed networks or ones for which we are not allowed to back-track.

**INVESTIGATION 3**

Many suburbs of Adelaide are laid out as rectangular grids. Suppose we start in one corner of a suburb and wish to travel to the opposite corner. A property of the rectangular grid is that all paths from the start S to the finish F have equal length so long as we never move away from the finish F. But how many of these paths are there?

**What to do:**

1 For each network below, how many ‘shortest’ paths are there from the start S to the finish F?
In ‘number of paths’ problems we are interested in finding the total number of paths which go from one node to another node without backtracking. Each connection on the graph allows traffic in one direction only.

Consider the directed network:

How many paths are there from A to B?

**Step 1:** From A, there are two possible nodes we can go to. There is one possible path to each of them so we write 1 near each.

**Step 2:** We then look at the next step in the journey. At each successive node we add the numbers from the previous nodes which lead to that point.

**Step 3:** Repeat Step 2 until we finish at B. So, there are 6 possible paths for getting from A to B.

**Note:**
- If we must avoid a given node, start by putting a zero (0) at that node.
- If we must pass through a particular node, start by “avoiding” nodes that do not allow passage through it.
Example 4

Alvin (A) wishes to visit Barbara (B) across town. Alvin’s other friend Sally lives at the intersection S. Find how many different pathways are possible in going from Alvin’s place to Barbara’s place:

a if there are no restrictions
b if Alvin must visit Sally along the way
c if Alvin must avoid Sally in case he gets stuck talking.

Starting from A, we write numbers on the diagram indicating the number of paths from the previous vertices. For example, to get to P we must come from S or T. So, \(6 + 4 = 10\).

The total number of paths is 35.

We must go through S, so there are some vertices which are no longer useful to us. We start by marking the places we cannot go with zeros.

The total number of paths from A to B is 18.

As S must be avoided, the total number of paths is 17.

EXERCISE 7C.1

In each question, assume that back-tracking is not allowed.

1 Calculate the number of paths from S to F in each of the following networks:

a

b

c

d
A shortest path problem is a problem where we must find the shortest path between one vertex (or node) and another in a network. The shortest path does not necessarily imply that all vertices are visited. They need not be.

Note also that by ‘shortest’ path we are not always referring to the path of shortest distance. We may also be referring to the quickest route that takes the shortest time, or the cheapest route of lowest cost.

A typical example is the problem of travelling from Adelaide to Mt Gambier. A network of the road route is shown alongside. The distances between towns are shown in kilometres.

Which route from Adelaide to Mt Gambier has the shortest distance?
Find the answer by trial and error.

**SHORTEST PATH PROBLEMS**

A shortest path problem is a problem where we must find the shortest path between one vertex (or node) and another in a network.

The shortest path does not necessarily imply that all vertices are visited. They need not be.

Note also that by ‘shortest’ path we are not always referring to the path of shortest distance. We may also be referring to the quickest route that takes the shortest time, or the cheapest route of lowest cost.

A typical example is the problem of travelling from Adelaide to Mt Gambier. A network of the road route is shown alongside. The distances between towns are shown in kilometres.

Which route from Adelaide to Mt Gambier has the shortest distance?
Find the answer by trial and error.

**INVESTIGATION 4  SHORTEST PATH BY ‘TRIAL AND ERROR’**

Eric has a choice of several paths to walk to school. The arcs of the directed network not only show the direction he must follow to take him closer to the school but also the time it takes (in minutes) to walk that section of the journey.

**What to do:**

1. How many different paths are possible for Eric to travel to school?
2. Describe the shortest path that he may travel from home to school.
3. What is the minimum time in which he can walk to school?
INVESTIGATION 5  SHORTEST PATH BY ‘TIGHT STRING’

Note: You will require string to cut into varying lengths for this investigation.

What to do:

Construct the network shown using lengths of string proportional to the lengths marked on the edges, for example the value 5 could be 5 cm in length, the value 4 could be 4 cm.

Tie the string at the nodes where edges meet.

While holding your network at the knots representing the vertices at either end of the path (A and E in this case), carefully pull the string tight.

The path that has been pulled tight is the shortest path.

EXERCISE 7C.2

1 Several street networks for students travelling from home to school are shown below. The numbers indicate the times in minutes. In each case, describe the quickest path and give the minimum travelling time.

2 George has just arrived at Heathrow airport in London. Can you help him work out the cheapest way to get to Epping using the Tube and overground trains? Prices for each section of the route are given in pounds sterling.

Although the trial and error and tight string methods are valid ways to find shortest paths, they become incredibly time-consuming when the networks are large and complicated.

The mathematician Dijkstra devised a much more efficient algorithm for large networks.
DIJKSTRA’S SHORTEST PATH ALGORITHM

**Step 1:** Assign a value of 0 to the starting vertex. Draw a box around the vertex label and the 0 to show the label is permanent.

**Step 2:** Consider all unboxed vertices connected to the vertex you have just boxed. Label them with the minimum weight from the starting vertex via the set of boxed vertices.

**Step 3:** Choose the least of all of the unboxed labels on the whole graph, and make it permanent by boxing it.

**Step 4:** Repeat steps 2 and 3 until the destination vertex has been boxed.

**Step 5:** Back-track through the set of boxed vertices to find the shortest path through the graph.

At each stage we try to find the shortest path from a given vertex to the starting vertex. We can therefore discard previously found shortest paths as we proceed, until we have obtained the actual shortest path from start to finish.

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**Example 5**

Find the shortest path from A to E in the network alongside.

**Step 1:** Assign 0 to the starting vertex and draw a box around it.

**Step 2:** The nodes B and D are connected to A. Label them with their distances from A.

**Step 3:** The unboxed vertex with the smallest label is D. Draw a box around it.

**Step 2:** Consider the unboxed nodes connected to D.
D to C has length 2, so we label C with \(3 + 2 = 5\).
D to E has length 7, so we label E with \(3 + 7 = 10\).

**Step 3:** The unboxed vertex with the smallest label is B. Draw a box around it.
Step 2: Consider the unboxed nodes connected to B. 
For C we would have $4 + 2 = 6$. This is more than 5 so the label is unchanged.
For E we have $4 + 5 = 9$, so we replace the label with 9.

Step 3: The unboxed vertex with the smallest label is C. Draw a box around it.

Step 2: The only unboxed node connected to C is E. $5 + 3 = 8$, and this is less than 9 so we change the label.

Step 3: The unboxed vertex with the smallest label is the finishing vertex E.

We now backtrack through the network to find the shortest path: $E \leftarrow C \leftarrow D \leftarrow A$.
So, the shortest path is $A \rightarrow D \rightarrow C \rightarrow E$ and its length is 8 units.

Note: 
- In practice, we do not usually write out all of the steps and numerous diagrams. These have only been included to help you understand the procedure.
- Suppose in Example 5 above that we are forced to pass through B on the way to E. To solve this problem we now find the minimum length from A to B and then add to it the minimum length from B to E.

Example 6

For the network alongside, find the shortest path from A to O that passes through M. What is the length of this path?

Since we must pass through M, we find the shortest path from A to M and then the shortest path from M to O.

Working backwards we get $O \leftarrow N \leftarrow M \leftarrow H \leftarrow G \leftarrow F \leftarrow A$.
$\therefore$ the shortest path is $A \rightarrow F \rightarrow G \rightarrow H \rightarrow M \rightarrow N \rightarrow O$ with length $7 + 6 = 13$ units.
EXERCISE 7C.3

1. Find the shortest path from A to Z and state its value:
   a. 
   b. 
   c. 
   d. 

2. Consider again the problem on page 375. Find the shortest route from Adelaide to Mt Gambier using the routes shown.

3. The time to run various tracks in an orienteering event is shown. Times are in minutes.
   a. Find the quickest way to get from P to Q.
   b. Which is the quickest way if the track PS becomes impassable due to a flood?

4. Solve Opening Problem question 1b for the quickest route between Dien’s house and Phan’s house.

5. A construction company has a warehouse at W in the diagram shown. Often several trips per day are made from the warehouse to each of the construction sites. Find the shortest route to each of the construction sites from the warehouse. Distances are in km.
6 The network alongside shows the connecting roads between three major towns, A, B and C. The weights on the edges represent distances, in kilometres.

a Find the length of the shortest path between towns A and C that goes through town B.

b Find the length of the shortest path between towns A and C.

7 A salesman is based in city T. The table shows the airfares in dollars for direct flights between the cities.

<table>
<thead>
<tr>
<th>T</th>
<th>U</th>
<th>V</th>
<th>W</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>80</td>
<td>–</td>
<td>30</td>
<td>–</td>
<td>80</td>
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<tr>
<td>U</td>
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<td>W</td>
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<tr>
<td>Y</td>
<td>80</td>
<td>50</td>
<td>60</td>
<td>50</td>
<td>–</td>
</tr>
</tbody>
</table>

a Put the information onto a network diagram.

b Determine the minimum cost of flights from T to each of the other cities.

c Determine the cheapest cost between T and Y if he must travel via V.

**DISCUSSION**

**SHORTEST PATH PROBLEMS**

What other situations involving ‘shortest’ paths can you think of?

**LONGEST PATH PROBLEMS**

In some situations we may need to find the longest path through a network. This may be to maximise profits or points collected on the way. Clearly we cannot allow back-tracking in the network since otherwise we could continually go to and from between nodes to increase the length forever. We can therefore only find longest paths if the network is directed or if we are not allowed to return to a node previously visited.

Unfortunately, we cannot find longest paths by simply changing “minimum” to “maximum” in Dijkstra’s Shortest Path Algorithm. There are other more complex algorithms that can be used, but they are beyond the scope of this course. We therefore present examples to be solved by trial and error.

**EXERCISE 7C.4**

1 A hardware salesman needs to travel from Adelaide to Mt Gambier for a meeting. He has six hours to make the journey, which gives him enough spare time that he can visit some clients along the way, but not enough that we can backtrack to pick up other routes. Based on previous sales he estimates the amount of money (in hundreds of dollars) he can earn along each section of the route. Which way should the salesman go to maximise his earnings?
2 The network shows the time in minutes to drive between intersections in a street network.

a Determine the route from S to M which would take the shortest time.

b Determine the route from S to M which would take the longest time.

SHORTEST CONNECTION PROBLEMS

In shortest connection problems we seek the most efficient way of joining vertices with edges so that paths exist from any vertex to any other vertex. Vertices do not need to be linked directly to each other, but they all must be somehow linked.

Situations requiring such a solution include computers and printers in a network (as in Opening Problem question 2), connecting a set of pop-up sprinklers to a tap, or connecting a group of towns with telephone or power lines.

You should recall from the beginning of this chapter that a spanning tree is a simple graph with no circuits which connects all vertices. The optimum solution requires a spanning tree for which the sum of the lengths of the edges is minimised. We call this the minimum length spanning tree.

INVESTIGATION 6

A school has decided to add four new drinking fountains for its students. The water is to come from an existing purification system at point A; the points B, C, D and E mark the locations of the new fountains. Clearly all fountains must be connected to the purification system, though not necessarily directly.

What to do:

1 Using the scale on the diagram, measure the length of pipe required for all fountains to be connected directly to the purification system.

2 By trial and error, find the minimum length spanning tree for the network which gives the shortest length of pipe to complete the project. What is this length of pipe?

EXERCISE 7C.5

Use trial and error to solve the following shortest connection problems:

1 a

b
Joseph the electrician is doing the wiring for a house extension. The new room is to have two powerpoints and a light in the centre of the ceiling. The numbers on the diagram indicate the distances in metres of cable needed between those points. What is the shortest length of cable required?

Just like the shortest path problem, the shortest connection problem becomes very time-consuming and awkward for large networks. However, we can again use algorithms to rapidly solve the problem.

**PRIM’S ALGORITHM**

In this algorithm we begin with any vertex. We add vertices one at a time according to which is nearest. For this reason Prim’s algorithm is quite effective on scale diagrams where edges are not already drawn in. It also has the advantage that because we add a new vertex with each step, we are certain that no circuits are formed. The algorithm is:

1. **Step 1:** Start with any vertex.
2. **Step 2:** Join this vertex to the nearest vertex.
3. **Step 3:** Join on the vertex which is nearest to either of those already connected.
4. **Step 4:** Repeat until all vertices are connected.
5. **Step 5:** Add the lengths of all edges included in the minimum length spanning tree.

### Example 7

Consider the example of connecting the school fountains in **Investigation 5**.

No distances between the purification system and the fountains were given, but the diagram was drawn to scale.

**Scale** 1 : 5000

**Step 1:** We choose to start at E.

**Step 2:** Clearly D is closest to E so we include DE in the solution.

(Length DE is 123 m using the scale diagram.)

**Step 3:** Consider distances from D and E. B is closest to D so we include BD in the solution.

(Length BD is 62 m.)
Step 4: Consider distances from B, D or E. A is the next closest vertex (to B), so we include AB in the solution. (Length AB is 75 m.)

Consider distances from A, B, D or E. C is the only vertex remaining. It is closest to B, so we include BC in the solution.
(Length BC is 140 m.)
All vertices are connected so we stop.

Step 5: The minimum length is 400 m.

NEAREST NEIGHBOUR ALGORITHM

In this algorithm we start with all edges of the network in place. We remove the longest edges one by one without disconnecting the graph until only the minimum length spanning tree remains. The algorithm is:

Step 1: Locate the longest length edge which can be removed without disconnecting the graph.
Step 2: Remove this edge.
Step 3: Repeat Steps 1 and 2 until the removal of any more edges will disconnect the graph.
Step 4: Add the lengths of all edges in the minimum length spanning tree.

Example 8

Use the Nearest Neighbour algorithm to find the minimum length spanning tree for the graph alongside.

The longest length edge is DG (length 15), so this is removed first.
The next longest edge is FG (length 14) and it can also be removed without disconnecting the graph.
There are two edges of weight 13. The edge that joins AH can be removed.
The edge EF cannot be removed because that would disconnect node F.
There are two edges of weight 12. The edge CD can be removed, but EG cannot be removed without disconnecting G and H.
DISCUSSION

MINIMUM CONNECTION ALGORITHMS

Discuss the advantages and disadvantages of Prim’s algorithm and the Nearest Neighbour algorithm. What determines which is better for a given problem?

Hint: One algorithm adds edges each step; the other removes edges each step.

For each of the following problems, use whichever algorithm is most appropriate to help find the solution:

3 Solve the Opening Problem question 2 for the minimum length of cable required for the computer network. Assume the lengths given are in metres.
4 Broadband cabling is to be installed to six locations. The distances, in metres, between each of these locations is shown alongside.

Find the plan that will connect all the locations for the least cost. How many metres of cable would be laid for this plan?

5 The diagram shows the distances (in km) between towns in an Alpine area. After heavy snow falls the road authorities wish to connect towns as quickly as possible by clearing the minimum distance of road.

a Find the roads that should be cleared and the minimum total length.

b Workers report that due to the risk of avalanche, it is too dangerous to clear route AC. What is the best plan now?

6 An aircraft company is proposing to set up a network of flights between the South Australian towns shown. The distances shown are flight distances of possible routes.

a Find the shortest length for the network of flights and illustrate the minimum length spanning tree. Comment on your answer.

b Reconsider the problem if it is essential that there are flights between Adelaide and Port Lincoln and Adelaide and Whyalla.

7 When Venice is flooded the authorities set up a network of raised platforms for pedestrians to walk on. If they connect the tourist attractions shown with the minimum length of platform possible, which walkways do they raise platforms over? Distances are given in metres.
A secure communication network is to be made between 12 towns as shown below. The weights on the edges represent the distances between towns, in kilometres.

If the cost of making the network is proportional to distance, find the connection that minimises the cost.

**THE CHINESE POSTMAN PROBLEM**

This problem was posed by the Chinese mathematician Kwan Mei-Ko:

A postman has to leave from a depot, travel along a set number of roads to deliver the mail, then return to the depot. How should he do this so he travels the minimum distance?

In effect, the postman has to travel along every edge of a weighted network. The edges are the roads and the nodes are their intersections. The weights indicate the length of each road.

Ideally, the postman would like to travel down every road only once. Such a delivery route is an Eulerian circuit, since we start and finish at the same vertex, and every edge is travelled exactly once. However, we know this is only possible if the degree of every vertex is even.

If an Eulerian circuit does not exist, then the postman needs to choose which edges to travel twice such that the extra distance travelled is minimised. He has to travel twice between whichever nodes have odd degree. So, we end up with a shortest path problem to solve.

**Example 9**

A council worker has to mark lines in the middle of the roads shown alongside, starting and ending at the depot. The weights on the graph indicate distances in kilometres.

Find the minimum distance that the marker can travel and a possible route he could follow.

All the vertices on the network graph are of even degree so an Eulerian circuit exists. This means that it is possible to travel all the edges of the graph exactly once, starting and finishing at the same point.

The minimum distance that he has to travel will be the sum of all the edges:

\[ 60 + 40 + 45 + 50 + 45 + 50 + 80 + 100 + 70 = 540 \text{ kilometres.} \]

A possible route would be depot-A-B-A-C-B-C-D-E-depot.
Solve the Chinese Postman Problem for the weighted graph shown, starting and finishing at D.

The degrees of vertices B and F are odd, so an Eulerian circuit does not exist.

We need to walk twice between B and F, so we use the shortest path algorithm to do this in the most efficient way:

The most efficient route is therefore to traverse $B \rightarrow E \rightarrow F$ twice.

The minimum distance travelled is the sum of all lengths in the diagram, plus 12, i.e., $57 + 12 = 69$.

A possible route is: D-C-B-E-F-A-B-E-F-G-H-E-D

Notice how B-E-F is used twice.

EXERCISE 7C.7

1 Solve the Chinese Postman Problem for each of the following networks:

2 Consider the network of roads joining towns L, M, N and O. Distances are shown in km.

The district council depot is in town L. The council wishes to inspect all the roads linking the towns, returning to L in the least possible distance.

Find the minimum distance to be travelled during the inspection and a possible route.
3 A leak exists somewhere in the sewerage network shown. An inspector needs to walk the length of all tunnels, entering and leaving through the manhole at O. For obvious reasons he wants to walk the least distance possible. Find the shortest distance that he can walk and state the path that leads to this shortest distance.

4 A parkland area has a central square (O) with paths arranged as shown.
Find the circuit of smallest length that enables the grounds staff to weed all paths. How long is this circuit?

5 A travel company offers a tour of cities of England for Australian visitors. They claim to offer scenic routes along all of the major connecting roads shown.

M is Manchester Y is York
C is Cambridge N is Nottingham
L is London O is Oxford

a Which road(s) need to be travelled twice to achieve this?
b What is the minimum road distance travelled? (Distances shown are in km.)

6 Solve Opening Problem question 3.

7 The malls in a shopping area are represented by the network shown. A security patrol must travel along each mall starting and finishing at S. The numbers represent the times (in mins) to walk each section.

a State the route for the patrolling officer which takes the minimum time.
b If she starts one round at 11:53 am, what time will she finish the round?
c How much earlier would she finish if mall AC is closed?

8 A council in outback South Australia plans to inspect all the local roads once every month. Plan a suitable route to enable this inspection, starting and finishing at O. Distances are given in km.

a Suggest a route, indicating roads that need to be travelled twice.
b Determine the distance that must be travelled.
c How does the situation change if extra roads are constructed between E and F and F and C? E to F is 30 km and F to C is 35 km.

THE TRAVELLING SALESMAN PROBLEM (TSP)

A travelling salesman leaves his home village, visits all of the villages in the neighbourhood, then returns home. How should he do this so that the distance he travels is minimised?

The salesman has to travel along the edges of a network, starting and finishing at the same node, and visiting every other node in between.

The problem is equivalent to finding the shortest Hamiltonian circuit on a complete graph. No algorithm exists that will efficiently solve the TSP for large networks. It therefore remains one of the great unsolved problems of pure mathematics.

However, for small networks we can solve the TSP relatively easily using an exhaustive search. This means we try all possibilities and choose the shortest. In practice this actually requires calculating only half the total number of possibilities, since we do not need to recalculate the reverse of a given route.

Example 11

Solve the TSP for the network alongside, assuming the salesman lives in village A. Distances are given in kilometres.

We find all of the Hamiltonian circuits in the graph that start and finish at A, and compare their total lengths. These are:

- ABCDA: $35 + 38 + 21 + 12 = 106$
- ABDCA: $35 + 23 + 21 + 33 = 112$
- ACBDA: $33 + 38 + 23 + 12 = 106$
- ADCBA: $12 + 21 + 38 + 35 = 106$
- ACDBA: $33 + 21 + 23 + 35 = 112$
- ADBCBA: $12 + 23 + 38 + 33 = 106$

Note that the three cycles on the right are simply those on the left in reverse order, so we need not have shown these calculations.

The minimum solution to the TSP is 106 km, using any of the four routes listed.
EXERCISE 7C.8

1 Solve the TSP for the following graphs, assuming the salesman lives in village O. All distances are given in kilometres.

![Graph a](image1)

![Graph b](image2)

2 Solve Opening Problem question 4.

3 An icecream van leaves a depot and stops at six spots during the afternoon. The map shows the locations and the times for possible routes between them (in mins). What is the fastest route?

![Icecream van routes](image3)

4 A postal van is to collect mail from mail boxes at various locations in a suburb. The numbers given indicate the times between boxes in minutes. What is the fastest route for collecting the mail?

![Postal van routes](image4)

D MODELLING WITH NETWORKS

In the previous section we considered a range of network problems and techniques for solving them. In this section we consider real-life, practical problems that can be solved by networks. The following projects will help you distinguish between the types of network problem and thus choose which solutions technique to apply.

**PROJECT IDEAS**

**A Pamphlet Drop**

Choose a map from a street directory. Define the area where you wish to deliver the pamphlets. Decide whether you will work in pairs or singly. One of each pair could walk down opposite sides of the street. If you work singly, most streets will probably need to be traversed twice: up one side and down the other. Draw a network of the area. Estimate distances to an appropriate accuracy using the scale on the map. Use your knowledge of networks and any other ideas to design the best way to carry out the pamphlet drop in the area determined.

**B Tourist Routes**

Choose several sites of interest around the city or town in which your school is located, or else sites in your state that would be attractive to tourists. Determine possible routes between them and estimate distances along these routes.
Plan a tour that visits each of the sites of interest. It should visit every site just once and cover the smallest possible distance.

**C Routes to School**

Draw a network showing the major routes from your home to school. You may use different modes of transport along different edges on your plan.

Estimate distances between nodes using a map or street directory.

Estimate times between nodes by trialling them. Use network ideas to establish the *shortest* and *quickest* routes between home and school. Remember these are not necessarily the same.

**D Watering Plan**

Consider possible positions for sprinklers for an area of lawn at home or at school.

Locate the source of water and plan the laying of pipes to bring water to the sprinklers so that the cost of piping is minimised.

---

**E INTERPRETING INFORMATION WITH MATRICES**

A matrix is a rectangular array of numbers. The plural of matrix is matrices.

Matrices can be used to store information about networks. They are particularly useful for large and complex networks.

Computers and graphic calculators use matrices for many tasks, so they are very useful tools for dealing with networks.

The matrix used to store a network is called an adjacency matrix.

**THE ADJACENCY MATRIX**

In the network below we see the possible paths between the houses of five friends.

These paths can be displayed in an adjacency matrix. Each vertex is represented by a row and a column in the matrix, so the number of rows and columns equals the number of vertices.

In general, if a network has \( n \) nodes its adjacency matrix representation is \( n \times n \).

An \( n \times n \) matrix is called a square matrix as it has the same number of rows as columns.

The entry in the row corresponding to vertex X and column corresponding to vertex Y is the number of edges leading directly from X to Y.

The adjacency matrix for the network above is therefore:

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 & 3 \\
1 & 0 & 1 & 0 & 1 \\
1 & 2 & 3 & 1 & 0
\end{bmatrix}
\]

One path from B to C

Two paths from E to B
UNDIRECTED NETWORKS

The five-friends introductory example is an undirected network and its adjacency matrix has some special features. Let us consider another example:

**Example 12**

Con, Wei and Sam live in different cities. They can communicate with each other by phone, fax, or email.

a Which people have:
   i internet access
   ii fax machines?

b Find the adjacency matrix of this undirected network.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>S</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td>From</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>S</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>W</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Consider the adjacency matrices for the undirected networks we have seen so far. Notice that in both cases:

- the matrices are **symmetric** about the leading diagonal. This will be true for all undirected networks.
- the leading diagonal consists of all 0s. Will this be true for all undirected networks?

Other features of networks are also easily seen from their matrices.

For example:

- a row of zeros corresponds to an **isolated vertex**
- a non-zero number on the leading diagonal shows a **loop**.
Adjacency matrices can be used to identify topologically equivalent networks. Remember these have the same structure but may look quite different.

For example, the two networks below are topologically equivalent, but at a glance this is far from obvious:

However, they have the same adjacency matrix:

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

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However, they have the same adjacency matrix:

$$
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
$$

**EXERCISE 7E.1**

1. Construct an adjacency matrix for each of the following networks:
2 Construct a network for each of the following adjacency matrices:

\[
\begin{align*}
\text{a} & : \begin{bmatrix}
1 & 2 & 1 & 1 \\
2 & 1 & 2 & 0 \\
1 & 2 & 1 & 1 \\
1 & 0 & 1 & 1
\end{bmatrix} \\
\text{b} & : \begin{bmatrix}
0 & 2 & 1 & 1 \\
2 & 0 & 2 & 0 \\
1 & 2 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix} \\
\text{c} & : \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\end{align*}
\]

3 Which of the following adjacency matrices represents an undirected network with an isolated vertex?

\[
\begin{align*}
\text{A} & : \begin{bmatrix}
1 & 1 & 2 & 0 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix} \\
\text{B} & : \begin{bmatrix}
1 & 2 & 1 & 1 \\
2 & 0 & 2 & 0 \\
1 & 2 & 1 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix} \\
\text{C} & : \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 2 & 0 \\
1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

4 The adjacency matrix for a network is given alongside. Which one of the following would not be true?

A There are loops on two of the vertices.
B One of the vertices is isolated.
C There are eight edges on this network.
D There are two edges joining one pair of vertices.
E There are 4 vertices on this network.

\[
\begin{align*}
\text{A} & : \begin{bmatrix}
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
\text{B} & : \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
\text{C} & : \begin{bmatrix}
2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
\text{D} & : \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

5 Use adjacency matrices to check if the following networks are topologically equivalent:

\[
\begin{align*}
\text{a} & \quad \text{and} \quad \text{b}
\end{align*}
\]

HISTORICAL NOTE

The famous ‘Four Colour Map problem’ states that “any map drawn on a plane can be coloured in four colours where no two adjacent regions have the same colour”. Arthur Cayley (1821-95) was one of the inventors of matrices and was one of the first to attempt to solve the ‘Four Colour Map problem’. The ‘Four Colour Map problem’ was eventually proven in 1977 using a computer in a demonstration that cannot be executed by hand. It took mathematicians 124 years to solve the problem and the construction is all in network format.
DIRECTED NETWORKS

If a network has directed arcs then the number of connections from A to B will not always equal the number of connections from B to A. The adjacency matrix will therefore not be symmetrical.

Example 14

At a new skifield there are two lifts to take the skiers from the kiosk to the A slope and three lifts taking them from the kiosk to the B slope. One lift takes them from the A slope to the higher B slope. Naturally the skiers do not return to the kiosk using the lifts because they ski back to the kiosk. Construct an adjacency matrix for the ski lifts.

The network diagram is:

To

From

<table>
<thead>
<tr>
<th></th>
<th>K</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
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<td>2</td>
<td>3</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The adjacency matrix is:

$$
\begin{bmatrix}
0 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
$$

We can also discover features of directed networks from their adjacency matrices. For the network in Example 14, notice that:

- The last row contains only zeros. This is because no directed arcs leave node B. Node B is called a sink.
- The first column contains only zeros. This is because directed arcs only leave node K. Node K is called a source.

EXERCISE 7E.2

1 Write down adjacency matrices for the following networks. Indicate nodes which are sources and sinks.

a

b
2 Construct networks for the following adjacency matrices. Indicate nodes which are sources and sinks.

\[
a = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]
\[
b = \begin{bmatrix}
0 & 0 & 2 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

**DOMINANCE MATRICES**

Directed networks can be used to display sporting results and other situations of dominance of one individual over another. They are particularly good at displaying results of round-robin tournaments such as in football or netball.

Consider a tri-nations rugby tournament in which Australia beats both New Zealand and South Africa, and New Zealand beats South Africa. We can display this as a directed network:

![Diagram of rugby tournament network]

This network has the adjacency matrix:

<table>
<thead>
<tr>
<th>loser</th>
<th>A</th>
<th>NZ</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>winner</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Australia</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>New Zealand</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>South Africa</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

and in such a context we call it a **dominance matrix**.

**EXERCISE 7E.3**

1. In a Superleague netball challenge: Melbourne Phoenix beat Sydney Swifts, Adelaide Thunderbirds and Melbourne Kestrels; Adelaide Thunderbirds beat Sydney Swifts and Melbourne Kestrels; Sydney Swifts beat Melbourne Kestrels.
   - a) Represent these results as a directed network.
   - b) Write down the corresponding dominance matrix.

2. The final results of a Tennis All Stars round-robin tournament were:
   Bjorn Borg beat Jimmy Conners and John MacEnroe; Jimmy Conners beat Ivan Lendl and John MacEnroe; Ivan Lendl beat Bjorn Borg; John MacEnroe beat Ivan Lendl.
   - a) Display the results as a directed network.
   - b) Write down the corresponding dominance matrix.
3 The results of an interclub handball competition are described by the dominance matrix

\[
\begin{array}{cccccc}
\text{loser} & R & C & G & P & H \\
\text{Rockets} & 0 & 0 & 1 & \frac{1}{2} & 1 \\
\text{Canons} & 1 & 0 & 0 & 1 & 0 \\
\text{Gunners} & 0 & 1 & 0 & 0 & 1 \\
\text{Pistols} & \frac{1}{2} & 0 & 1 & 0 & 1 \\
\text{Harpoons} & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

\( \text{winner} \)

a Explain why there are 0s along the leading diagonal.

b Suggest what the \( \frac{1}{2} \)s would represent.

c Construct a network to display the results.

d Which teams should play off in the final?

---

**A COMPUTER PACKAGE TO HANDLE NETWORKS**

Mathematical computer packages such as MATHEMATICA® can handle network problems. Below we see the instruction (in bold type) and output for the MATHEMATICA® representation of \( K_5 \), the complete graph on 5 vertices. Beside it is the instruction for creating the adjacency matrix.

\[
\text{In}[1]: = \text{ShowGraph}[K[5]]; \quad \text{In}[2]: = \text{TableForm}[\text{Edges}[k[5]]]; \\
\text{Out}[2]/\text{TableForm} = \\
\begin{array}{cccccc}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
\end{array}
\]

The advantages of using matrix forms include the obvious extension to very large graphs which are difficult or impossible to draw in full. A computer can store the corresponding large matrix efficiently as most of the entries are usually zeros.

In packages such as MATHEMATICA® the computer will also quickly find and output many features like possible **Eulerian circuits** and **Hamiltonian paths**.

---

**FURTHER MATRICES**

The use of matrices in network problems is only one minor application of matrices. Matrices have many uses. Spreadsheets displaying rectangular arrays of stock in hand numbers, costings, budgets etc, are displaying matrices and they may be very large indeed.

You have been using matrices for many years without realising it.

For example:

<table>
<thead>
<tr>
<th>Goals</th>
<th>Behinds</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crows</td>
<td>15</td>
<td>9</td>
</tr>
<tr>
<td>Power</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ingredients</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>sugar</td>
<td>3 tspn</td>
</tr>
<tr>
<td>flour</td>
<td>1 cup</td>
</tr>
<tr>
<td>milk</td>
<td>150 mL</td>
</tr>
<tr>
<td>salt</td>
<td>2 pinches</td>
</tr>
</tbody>
</table>
In general:

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

It is usual to put square or round brackets around a matrix.

Consider these two items of information:

### Furniture inventory

<table>
<thead>
<tr>
<th></th>
<th>chairs</th>
<th>tables</th>
<th>beds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Unit</td>
<td>6</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>House</td>
<td>8</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

We could write them in detailed matrix form as:

\[
\begin{bmatrix}
B & 1 & \text{and} & C & T & B \\
J & 3 & \text{and} & F & 4 & 1 & 2 \\
E & 12 & \text{and} & U & 6 & 1 & 3 \\
C & 1 & & H & 8 & 2 & 4
\end{bmatrix}
\]

and if we can remember what makes up the rows and columns, we could write them simply as:

\[
\begin{bmatrix}
1 \\
3 \\
12 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 & 1 & 2 \\
6 & 1 & 3 \\
8 & 2 & 4
\end{bmatrix}
\]

we have 4 rows and 1 column and we say that this is a \(4 \times 1\) **column matrix**.

\[
\begin{bmatrix}
1 & 3 & 12 \\
1 & 3 & 12 \\
1 & 3 & 12
\end{bmatrix}
\]

has 1 row and 4 columns and is called a \(1 \times 4\) row matrix.

\[
\begin{bmatrix}
3 & 0 & -1 & 2 \\
4 & 1 & 2 \\
6 & 1 & 3 \\
8 & 2 & 4
\end{bmatrix}
\]

has 3 rows and 3 columns and is called a \(3 \times 3\) **square matrix**.

this element, 2, is in row 3, column 2.

**Note:**

- An \(m \times n\) matrix has \(m\) rows and \(n\) columns.
- \(m \times n\) specifies the **order** of a matrix.
**DISCUSSION**

Why do we agree to state rows first, then columns?

**EXERCISE 7F**

1. Write down the order of:
   - a \([ 6 \ 2 \ 3 ]\)
   - b \([ 3 \ ]\)
   - c \([ 3 \ 4 \ ]\)
   - d \([ 2 \ 0 \ 4 \ 5 \ 1 \ 0 ]\)

2. For the matrix \(\begin{bmatrix} 3 & 1 & 2 & 5 \\ 1 & 0 & 6 & 4 \\ 4 & 1 & 7 & 3 \end{bmatrix}\) what element is found in
   - a row 1, column 3
   - b row 3, column 1
   - c row 3, column 4?

**Example 15**

Peter goes shopping at store A to buy 2 loaves of bread at $2.35 each, 4 litres of milk at $1.75 per litre, and one 500 g tub of butter at $3.25.

- a Represent the quantities purchased in a row matrix.
- b Represent the costs in a column matrix.
- c If Peter goes to a different supermarket (store B) and finds that the prices for the same items are $2.25 for bread, $1.50 for milk, and $3.20 for butter, write one cost matrix which shows prices from both stores.

- a The quantities matrix is \(\begin{bmatrix} 2 & 4 & 1 \end{bmatrix}\)
- b The costs matrix is \(\begin{bmatrix} 2.35 \\ 1.75 \\ 3.25 \end{bmatrix}\)
- c The costs matrix is now:

\[
\begin{bmatrix}
2.35 & 2.25 \\
1.75 & 1.50 \\
3.25 & 3.20
\end{bmatrix}
\]

- Claude went to a hardware store to purchase items for his home construction business. He bought 6 hammers, 3 pinchbars, 5 screwdrivers, and 4 drill sets. The costs of the items were $12, $35, $3, and $8 respectively.

- a Construct a row matrix showing the quantities purchased.
- b Construct a column matrix showing the prices of the items in the appropriate order.
4. Seafresh Tuna cans are produced in three sizes: 100 g, 250 g and 500 g. In February they produced respectively: 2500, 4000 and 6500 cans in week 1; 3000, 3500 and 7000 cans in week 2; 3500, 3500 and 6500 cans in week 3; 4500 of each type in week 4.

a. Construct a $3 \times 4$ matrix to display production levels.

b. Construct a $4 \times 3$ matrix to display production levels.

5. A baker produces pies, pasties, sausage rolls and pizza slices over a long weekend. On Saturday she produced 30 dozen pies, 40 dozen pasties, 25 dozen sausage rolls and 20 dozen pizza slices. On Sunday she produced 25 pies, 50 dozen pasties, 30 dozen sausage rolls and 25 dozen pizza slices. On the Monday she made 30 dozen of all four items.

a. Construct a $3 \times 4$ matrix to display the items made.

b. Construct a $4 \times 3$ matrix to display the items made.

RESEARCH

On the packaging of most consumable food items you will find a matrix showing the ingredients and nutritional content. Collect three of these matrices for basically the same product (or similar product). Combine them into a single matrix.

In a brief report (50 words) outline how the information in the matrix could be used.

DISCUSSION

Discuss the usefulness of organising data in matrix form.

G

ADDITION AND SUBTRACTION OF MATRICES

ADDITION

Harry has three stores (A, B and C). His stock levels for TV sets, microwave ovens and refrigerators are given by the matrix:

\[
\begin{pmatrix}
23 & 41 & 68 \\
28 & 39 & 79 \\
46 & 17 & 62
\end{pmatrix}
\]

Some newly ordered stock has just arrived. For each store 30 TVs, 40 microwaves and 50 refrigerators must be added to stock levels.

His stock order is given by the matrix:

\[
\begin{pmatrix}
30 & 30 & 30 \\
40 & 40 & 40 \\
50 & 50 & 50
\end{pmatrix}
\]

Clearly the new levels are shown as:

\[
\begin{pmatrix}
23 + 30 & 41 + 40 & 68 + 30 \\
28 + 40 & 39 + 40 & 79 + 40 \\
46 + 50 & 17 + 50 & 62 + 50
\end{pmatrix}
\]
or
\[
\begin{bmatrix}
23 & 41 & 68 \\
28 & 39 & 79 \\
46 & 17 & 62
\end{bmatrix} + \begin{bmatrix}
30 & 30 & 30 \\
40 & 40 & 40 \\
50 & 50 & 50
\end{bmatrix} = \begin{bmatrix}
53 & 71 & 98 \\
68 & 79 & 119 \\
96 & 67 & 112
\end{bmatrix}
\]

So, to **add** two matrices they must be of the **same order** and then we simply add corresponding elements.

**SUBTRACTION**

If Harry’s stock levels were
\[
\begin{bmatrix}
39 & 61 & 29 \\
41 & 38 & 42 \\
50 & 27 & 39
\end{bmatrix}
\]
and his sales matrix for the week is
\[
\begin{bmatrix}
16 & 13 & 7 \\
21 & 17 & 20 \\
20 & 9 & 15
\end{bmatrix}
\]
what are the current stock levels?

It is obvious that we subtract corresponding elements.

That is
\[
\begin{bmatrix}
39 & 61 & 29 \\
41 & 38 & 42 \\
50 & 27 & 39
\end{bmatrix} - \begin{bmatrix}
16 & 13 & 7 \\
21 & 17 & 20 \\
20 & 9 & 15
\end{bmatrix} = \begin{bmatrix}
23 & 48 & 22 \\
20 & 21 & 22 \\
30 & 18 & 24
\end{bmatrix}
\]

So, to **subtract** matrices they must be of the **same order** and then we simply subtract corresponding elements.

---

### Example 16

**Self Tutor**

At the end of March, Harry’s stock matrix was:

<table>
<thead>
<tr>
<th></th>
<th>Store</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>TVs</td>
<td>28</td>
</tr>
<tr>
<td>microwaves</td>
<td>19</td>
</tr>
<tr>
<td>fridges</td>
<td>25</td>
</tr>
</tbody>
</table>

His stock order matrix in early April was:

\[
\begin{bmatrix}
20 & 37 & 23 \\
21 & 16 & 18 \\
15 & 24 & 30
\end{bmatrix}
\]
and his sales during April were:

\[
\begin{bmatrix}
31 & 28 & 26 \\
20 & 31 & 19 \\
25 & 17 & 32
\end{bmatrix}
\]

What is Harry’s stock matrix at the end of April?

<table>
<thead>
<tr>
<th></th>
<th>At the beginning of April the stock matrix was:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[28 &amp; 19 &amp; 15 ] + [20 &amp; 21 &amp; 15 ] = [48 &amp; 40 &amp; 17 ]</td>
</tr>
<tr>
<td></td>
<td>[13 &amp; 24 &amp; 30 ] + [37 &amp; 16 &amp; 24 ] = [50 &amp; 40 &amp; 40 ]</td>
</tr>
<tr>
<td></td>
<td>[17 &amp; 40 &amp; 15 ] + [28 &amp; 21 &amp; 24 ] = [50 &amp; 40 &amp; 40 ]</td>
</tr>
<tr>
<td></td>
<td>[40 &amp; 40 &amp; 40 ] - [31 &amp; 20 &amp; 25 ] = [9 &amp; 20 &amp; 15 ]</td>
</tr>
<tr>
<td></td>
<td>[40 &amp; 40 &amp; 40 ] - [31 &amp; 20 &amp; 25 ] = [9 &amp; 20 &amp; 15 ]</td>
</tr>
<tr>
<td></td>
<td>[40 &amp; 40 &amp; 40 ] - [31 &amp; 20 &amp; 25 ] = [9 &amp; 20 &amp; 15 ]</td>
</tr>
</tbody>
</table>
EXERCISE 7G

1. At the end of April Harry’s stock matrix is
   \[
   \begin{bmatrix}
   17 & 22 & 14 \\
   20 & 9 & 21 \\
   15 & 23 & 8
   \end{bmatrix}
   \]
   a. What will be his stock order matrix if he wishes to have 50 of each item in each
      store at the start of May?
      \[
      \begin{bmatrix}
      22 & 17 & 28 \\
      19 & 22 & 30 \\
      41 & 7 & 44
      \end{bmatrix}
      \]
      What is his stock matrix at the end of May?
   b. During May, Harry’s sales matrix is
      \[
      \begin{bmatrix}
      22 & 17 & 28 \\
      19 & 22 & 30 \\
      41 & 7 & 44
      \end{bmatrix}
      \]
      a. What will be his stock order matrix if he wishes to have 50 of each item in each
         store at the start of May?
      b. During May, Harry’s sales matrix is
         \[
         \begin{bmatrix}
         22 & 17 & 28 \\
         19 & 22 & 30 \\
         41 & 7 & 44
         \end{bmatrix}
         \]
         What is his stock matrix at the end of May?

2. A restaurant served 72 men, 84 women and 49 children on Friday night. On Saturday
   night they served 86 men, 72 women and 46 children.
   a. Express this information in two column matrices.
   b. Use the matrices to find the totals of men, women and children served over the
      Friday-Saturday period.

3. On Tuesday Keong bought shares in five companies and on Friday he sold them.
   The details are:
   \[
   \begin{array}{|c|c|c|}
   \hline
   & \text{Cost price per share} & \text{Selling price per share} \\
   \hline
   A & $1.23 & $1.38 \\
   B & $22.15 & $22.63 \\
   C & $0.72 & $0.69 \\
   D & $3.75 & $3.68 \\
   E & $4.96 & $5.29 \\
   \hline
   \end{array}
   \]
   a. Find Keong’s i cost price column matrix ii selling price column matrix.
   b. What matrix operation is needed to find Keong’s profit/loss matrix?
   c. Find Keong’s profit/loss matrix.

4. During weekdays a video store finds that its average hirings are: 129 movies (VHS),
   176 movies (DVD) and 129 video/computer games. On the weekends the average figures
   are: 288 DVD movies, 129 VHS movies and 132 games.
   a. Represent the data using two column matrices.
   b. Find the sum of the matrices in a.
   c. What does the sum matrix in b represent?

5. Cedrick’s stock matrix for items in his stores was:
   \[
   \begin{bmatrix}
   58 & 79 & 32 & 44 \\
   62 & 49 & 68 & 53 \\
   117 & 129 & 89 & 144
   \end{bmatrix}
   \]
   His sales matrix for the next week was:
   \[
   \begin{bmatrix}
   26 & 38 & 40 & 39 \\
   50 & 42 & 51 & 38 \\
   87 & 90 & 38 & 100
   \end{bmatrix}
   \]
   and his new order matrix:
   \[
   \begin{bmatrix}
   50 & 60 & 60 & 70 \\
   40 & 50 & 40 & 50 \\
   70 & 80 & 80 & 70
   \end{bmatrix}
   \]
   Find Cedrick’s stock matrix at the end of this period.

6. Why do matrices need to be of the same shape in order to add or subtract them?
A matrix can be multiplied by a number (often called a **scalar** to produce another matrix. For example, in a block of 8 identical flats each flat has 2 tables, 6 chairs, 3 beds and 1 wardrobe.

If \( F = \begin{bmatrix} 2 \\ 6 \\ 3 \\ 1 \end{bmatrix} \) represents the furniture for one flat, then in terms of \( F \), the matrix representing the furniture in all flats can be written as \( 8F = \begin{bmatrix} 8 \times 2 \\ 8 \times 6 \\ 8 \times 3 \\ 8 \times 1 \end{bmatrix} \) or \( \begin{bmatrix} 16 \\ 48 \\ 24 \\ 8 \end{bmatrix} \).

In general, if a scalar \( k \) is multiplied by a matrix \( A \) the result is matrix \( kA \) obtained by multiplying every element of \( A \) by \( k \).

**EXERCISE 7H**

1. Freda’s usual weekly order for store A is: 30 dresses, 20 blouses and 24 suits. For store B it is: 40 dresses, 36 blouses and 30 suits.
   a. Write this information as a \( 2 \times 3 \) matrix.
   b. If Freda’s order is halved for next week, what will be her order matrix?
   c. If Freda decides to order for two weeks, what will be her order matrix?

2. Bob’s order for hardware items is shown in matrix form as
   \[
   H = \begin{bmatrix} 10 \\ 20 \\ 100 \\ 40 \end{bmatrix}
   \]
   Find the matrix if:
   a. Bob doubles his order
   b. Bob halves his order
   c. Bob increases his order by 50%.

3. Sarah sells dresses made by four different companies which we will call A, B, C and D. Her usual monthly order is:
   \[
   \begin{array}{cccc}
   & A & B & C & D \\
   skirt & 40 & 30 & 30 & 80 \\
   dress & 50 & 40 & 30 & 60 \\
   evening & 40 & 60 & 50 & 40 \\
   suit & 20 & 20 & 20 & 20 \\
   \end{array}
   \]
   Find her order if:
   a. she increases her total order by 20%
   b. she decreases her total order by 20%.

4. Colin’s stock matrix is \( M \). His normal weekly order matrix is \( N \) and his sales matrix is \( S \). Unfortunately this time Colin ordered twice. Which of the following will be his actual stock holding at the end of this period?
   \[
   \begin{align*}
   A &= M + N + S \\
   B &= M + N - S \\
   C &= M + 2N - S \\
   D &= M + N - 2S
   \end{align*}
   \]
MATRIX MULTIPLICATION

Suppose Peter needs to buy 2 wardrobes, 3 tables and 10 chairs and has two stores A and B from which he can make the purchases. He has decided, however, that he will buy the items from only one of these two stores, so that he can ask for bulk discount.

The matrix $\begin{bmatrix} W & T & C \end{bmatrix}$ could represent the numbers required of each type of furniture.

In store A a wardrobe costs $400, a table costs $300, and a chair costs $50.
In store B a wardrobe costs $500, a table costs $200, and a chair costs $65.

The costs matrix (in dollars) for store A and store B is therefore:

$\begin{bmatrix} W & T & C \end{bmatrix} = \begin{bmatrix} 2 & 4 & 300 \\ 4 & 5 & 200 \\ 5 & 6 & 65 \end{bmatrix}$

Notice that the total cost for store A items will be

$2 \times 400 + 3 \times 300 + 10 \times 50 = 2200$ dollars

and the total cost for store B items will be

$2 \times 500 + 3 \times 200 + 10 \times 65 = 2250$ dollars.

To do this using matrices, we want

$\begin{bmatrix} W & T & C \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 2200 \\ 2250 \end{bmatrix}$

orders:

$1 \times 3$ and $3 \times 2$ the same resulting matrix

To multiply two matrices $A$ and $B$ we make sure that the number of columns of $A$ matches the number of rows of $B$.

The product of $A$ and $B$ is written $AB$.

We multiply each member of a row of $A$ by each member of a column of $B$ and add these results. We do this for all possible rows and columns.

If $A$ is $m \times n$ and $B$ is $n \times p$ then $AB$ is $m \times p$.

Example 17

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$, find $AB$.

$A$ is $2 \times 3$ and $B$ is $3 \times 2$.

$AB$ is $2 \times 2$.

$AB = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 + 6 + 15 & 1 + 0 + 20 \\ 2 + 8 + 18 & 2 + 0 + 24 \end{bmatrix} = \begin{bmatrix} 22 & 21 \\ 28 & 26 \end{bmatrix}$
EXERCISE 7I

1 If \( A \) is \( 3 \times 2 \) and \( B \) is \( 2 \times 1 \):
   a explain whether \( AB \) can be found
   b find the shape of \( AB \)
   c explain why \( BA \) cannot be found.

2 a For \( A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \) and \( B = \begin{bmatrix} 5 & 6 \end{bmatrix} \), find \( BA \).
   b For \( A = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \) find i \( AB \) ii \( BA \).

3 Find:
   a \( \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \)
   b \( \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \)
   c \( \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix} \)
   d \( \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix} \)

4 At the local fair it costs \$3\) for a pony ride and \$5\) for a go-cart ride. On day 1 there are 43 pony rides and 87 go-cart rides. On day 2 there are 48 pony rides and 66 go-cart rides.
   a Write the costs in the form of a \( 2 \times 1 \) cost matrix \( C \).
   b Write down the \( 2 \times 2 \) numbers matrix \( N \).
   c Find \( NC \) and interpret your result.
   d Find the total income for both rides over the two days.

5 You and your friend each go to your local supermarkets \( A \) and \( B \) to price items you wish to purchase. You want to buy 1 leg of ham, 1 Christmas Pudding and 2 litres of cola. Your friend wants 1 leg of ham, 2 Christmas puddings and 3 litres of cola. The prices of these goods are:

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( B )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Leg of ham</strong></td>
<td>$43</td>
<td>$39</td>
</tr>
<tr>
<td><strong>Christmas pudding</strong></td>
<td>$7</td>
<td>$8</td>
</tr>
<tr>
<td><strong>Litre of cola</strong></td>
<td>$3</td>
<td>$4</td>
</tr>
</tbody>
</table>

a Write the requirements matrix \( R \) as a \( 3 \times 2 \) matrix.

b Write the prices matrix \( P \) as a \( 2 \times 3 \) matrix.

c Find \( PR \).

d What are your costs at \( A \) and your friend’s costs at \( B \)?

e Should you buy from \( A \) or \( B \)?
Roxanne buys 5 shirts, 3 skirts and 4 dresses costing $47, $25 and $68 respectively.

a The quantities matrix could be $3 \times 1$ or $1 \times 3$. Similarly, the costs matrix could be $3 \times 1$ or $1 \times 3$. However the total cost needs to be $1 \times 1$. How is this achieved with the quantities matrix $Q$ and the prices matrix $P$?

b What are $Q$ and $P$ and $QP$?

c What is the total cost?

### Using Technology for Matrix Operations

Matrices are easily stored in a graphics calculator.

For example, the matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{bmatrix}$ can be stored using these instructions:

#### Using a Texas Instruments TI-83

Press [MATRIX] to display the matrices screen, and use [►] to select the EDIT menu. This is where you define matrices and enter the elements.

Press 1 to select 1:[A]. Press ENTER 2 ENTER to define matrix $A$ as a $3 \times 2$ matrix.

Enter the elements of the costs matrix, pressing ENTER after each entry.

Press [SHIFT] [MODE] (QUIT) when you are done.

#### Using a Casio fx-9860G

Select Run:Mat from the main menu, and press [F1] (►MAT). This is where you define matrices and enter their elements.

To define matrix $A$, make sure Mat $A$ is highlighted, and press [F3] (DIM) 3 [EXE] 2 [EXE] [EXE].

Enter the elements of the costs matrix, pressing [EXE] after each entry.

Press [EXIT] twice to return to the home screen when you are done.
Now consider \( A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 2 & 0 \\ 3 & 8 \end{bmatrix} \) and \( C = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \).

We notice that \( A + B = \begin{bmatrix} 3 & 9 \\ 3 & 4 \\ 8 & 8 \end{bmatrix} \), but \( A + C \) cannot be found.

The following instructions will help you check these answers:

Using a Texas Instruments \textsc{TI-83}

Define matrices \( A, \ B \) and \( C \) using the method given on page 406.

To find \( A + B \), press \text{MATRIX} \ 1 \ to \ enter \ matrix \( A \), then \ + \ , then \text{MATRIX} \ 2 \ to \ enter \ matrix \( B \).

Press \text{ENTER} \ to \ display \ the \ result.

Attempting to find \( A + C \) in a similar manner will produce an error message, as \( A \) and \( C \) have different orders.

Using a Casio \textsc{fx-9860G}

Define matrices \( A, \ B \) and \( C \) using the method given on page 406.

To find \( A + B \), press \text{OPTN} \ F2 \ (MAT) \ F1 \ (Mat) \ \text{ALPHA} \ x, \theta, T \ (A) \ to \ enter \ matrix \( A \), then \ + \ , then \ F1 \ (Mat) \ \text{ALPHA} \ \log \ (B) \ to \ enter \ matrix \( B \).

Press \text{EXE} \ to \ display \ the \ result.

Attempting to find \( A + C \) in a similar manner will produce an error message, as \( A \) and \( C \) have different orders.

\textbf{Note:} \ Operations such as \( A - B, \ 2A, \ 3B, \ \text{and} \ 2A + 5B \) are now easily found.

Now consider finding \( \begin{bmatrix} 3 & 1 \\ \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 4 & 7 \end{bmatrix} \).

Using a Texas Instruments \textsc{TI-83}

Define matrix \( A = \begin{bmatrix} 3 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 5 & 6 \\ 4 & 7 \end{bmatrix} \) using the method given on page 406.

To find \( AB \), press \text{MATRIX} \ 1 \ to \ enter \ matrix \( A \), then \ \times \ , then \text{MATRIX} \ 2 \ to \ enter \ matrix \( B \).

Press \text{ENTER} \ to \ display \ the \ result.
Using a Casio fx-9860G

Define matrix  \( A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \) and  \( B = \begin{bmatrix} 5 & 6 \\ 4 & 7 \end{bmatrix} \)
using the method given on page 406.

To find  \( AB \), press \( \text{OPTN} \, \text{F2} \, (\text{MAT}) \, \text{F1} \, (\text{Mat}) \) \( x \, \theta \, T \) (A) to enter matrix A, then \( X \), then \( \text{F1} \, (\text{Mat}) \, \text{ALPHA} \, \text{log} \, (B) \) to enter matrix B.

Press \( \text{EXE} \) to display the result.

**EXERCISE 7J**

Using technology to answer all of these questions:

1. \( A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 0 & 5 \\ 2 & 1 & 6 \end{bmatrix} \) \( , \) \( B = \begin{bmatrix} 3 & 1 & 9 \\ 0 & 2 & 7 \\ 1 & 5 & 4 \end{bmatrix} \) \( , \) \( C = \begin{bmatrix} 1 & 5 & 6 \end{bmatrix} \) \( \) and \( D = \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix} \).

Use technology to find:

\( a \) \( 37A \) \( b \) \( 46B \) \( c \) \( 23A + 97B \) \( d \) \( 39A - 17B \)

\( e \) \( AC \) \( f \) \( CA \) \( g \) \( BD \) \( h \) \( DB \)

2. The selling prices \( \$ \) The costs incurred \( \$ \)
of five white goods items are \( A \) 238 \( \) by the store in \( A \) 164 \( \)
given by the \( S = \) \( C \) 529 \( \) items are given \( C = \) \( C \) 398 \( \)
matrix \( D \) 381 \( \) by the matrix \( D \) 306 \( \)
\( E \) 267 \( \) \( E \) 195 \( \)

\( a \) Find the profit matrix for the five items.

\( b \) Suppose 48 of item A, 67 of B, 103 of C, 89 of D and 114 of E are sold. Write down the corresponding \( 1 \times 5 \) sales matrix \( N \).

\( c \) Find the total profit made on the sale of these items using matrix multiplication.

3. \( \begin{array}{cccc} P & Q & R & S & T \\ 17 & 3 & 0 & 14 & 9 \\ 31 & 5 & 1 & 16 & 14 \\ 28 & 8 & 2 & 22 & 26 \\ 32 & 7 & 1 & 9 & 32 \\ 19 & 6 & 3 & 11 & 17 \\ 31 & 9 & 0 & 18 & 18 \end{array} \) is a sales matrix and \( R \) \( $22 \) is a price matrix.

Find the \( 6 \times 1 \) matrix for the total income of each day.

4. Four supermarkets owned by Jason sell 600 g cans of peaches. Store A buys them for $2.00 and sells them for $3.00; Store B buys them for $1.80 and sells them for $2.90; Store C buys them for $2.10 and sells them for $3.15; Store D buys them for $2.25 and sells them for $2.95.
a Set up two column matrices, one for costs $C$ and one for selling prices $S$.

b What is the profit matrix for each store?

c If A sells 367, B sells 413, C sells 519 and D sells 846 cans in one week, determine the total profit made on the cans during the week.

5 Sadi owns five stores. She keeps track of her six different sales items using the stock matrix $S$ alongside:

$$S = \begin{bmatrix} 1 & 42 & 24 & 26 & 52 & 39 & 68 \\ 2 & 38 & 27 & 32 & 60 & 50 & 68 \\ 3 & 57 & 27 & 40 & 58 & 40 & 73 \\ 4 & 39 & 30 & 38 & 47 & 39 & 79 \\ 5 & 62 & 31 & 32 & 71 & 52 & 80 \end{bmatrix}$$

a Noting that $S$ is a $5 \times 6$ matrix, should Sadi write down the profits matrix with order $6 \times 1$ or $1 \times 6$? Explain your answer.

b Find a matrix which shows the total profit for each store.

6 Paula writes a list of furniture items needed to refurbish her 8 flats, 5 apartments and 2 houses. This list is given alongside. Chairs cost $30, tables $150, lounges $450, wardrobes $550 and sideboards $700 each.

a Write the accommodation types as a $1 \times 3$ matrix $A$.

b Write the furniture items needed as a $3 \times 5$ matrix $N$.

c Find $AN$ and state the meaning of this matrix.

d Write the cost matrix $C$ in $5 \times 1$ form.

e Now find $ANC$ and state its meaning.

DISCUSSION

Imagine you were given question 6 above but without steps a, b, c and d. If you were asked to find the total cost of all the furniture, how would you have decided that matrix $ANC$ was the one to be found?

REVIEW SET 7

1 A network has seven vertices and is connected. What is the minimum number of edges it could have?

2 A complete network has five vertices. How many edges does it have?

3 What additional edge could be added to the given graph to make sure the resulting graph contains an Eulerian circuit?
4 Which of the following graphs is a spanning tree for the network shown below?

A
B
C
D
E

5 Which of the following graphs does not have an Eulerian Circuit?

A
B
C
D
E

6 Construct a network diagram for the project of renovating a room using the following tasks:

<table>
<thead>
<tr>
<th>Task</th>
<th>Prerequisite Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Remove carpet</td>
<td>-</td>
</tr>
<tr>
<td>B Sand timber floor</td>
<td>A</td>
</tr>
<tr>
<td>C Plaster walls</td>
<td>-</td>
</tr>
<tr>
<td>D Electrical work</td>
<td>-</td>
</tr>
<tr>
<td>E Paint room</td>
<td>A, B, C, D</td>
</tr>
<tr>
<td>F Seal floor</td>
<td>A, B, C, D, E</td>
</tr>
</tbody>
</table>

7 Determine the tasks for the following projects. List them in a precedence table and draw a network diagram to model each project.

a Changing a flat tyre on a bicycle.

b Making a chocolate cake.

c Tiling a bathroom.

d Preparing an evening meal of soup, roast, and dessert.

8 A vehicle is travelling from town P to town Q. The journey requires the vehicle to travel along a network linking suitable fuel stops. The cost of travel between these is shown on the network where the nodes represent fuel stops. Find the minimum cost for the trip.
9. Find the adjacency matrix for the network:

10. Find the minimum length spanning tree for the given network. Determine its length.

11. How many paths exist from A to B which pass through X?

12. How many paths go from P to R without passing through Q?

13. a. Does the given network have a Hamiltonian circuit?
b. Does it have an Eulerian circuit?
c. Write the adjacency matrix for the network.

14. Find the shortest path between the source and the sink.

15. How many paths exist from A to B?

16. Draw a network diagram to represent the roads between towns A, B, C, D and E if the following connections exist:

   Town A is 35 km from Town B and 12 km from Town E
   Town B is 20 km from Town C and 23 km from Town D
   Town C is 45 km from Town D.

17. Find the shortest paths in the following network from:
   a. B to G
   b. A to F
   c. H to C

18. Apply Dijkstra’s algorithm to find the shortest distance from start to finish in the following networks:

   a. [Diagram showing a network with nodes A to G and distances between them]
   b. [Diagram showing a network with nodes A to J and distances between them]
19 A telecommunications company wishes to lay the cable required to connect the six towns shown. The numbers indicate costs for each possible connection. Suggest how the towns should be connected.

20 A company is building the new Bigtown University. They have constructed new faculty buildings in a layout as shown. The minimum distance (in metres) between adjacent buildings is also shown. Draw a network which ensures all buildings are connected to the University computer network, but which minimises the amount of cable used. What is this length of cable?

21 A large department store wishes to link their cash registers so that stock information is available to all staff. The table below shows the distances (in metres) between each of the cash registers. Costs are calculated on two factors: cabling that costs $3.50 per metre, and labour that is charged at $45.00 per hour. It takes approximately 1 hour to lay 6 metres of cable. Suggest how the department store should link the terminals. Support your suggestion with calculations.

22 a A walker follows the route A-B-A-F-E-D-C-E-F-A. Why is this route not a Hamiltonian circuit?

b Write down a route which is a Hamiltonian circuit.

c The National Park Authority limits the number of people per day who can travel along each path. The traffic limits are shown. Note that a landslide has blocked the path CE at present. What is the maximum number of people per day who can travel from A to C along the paths?

23 A roadsweeper based at A must clean all of the roads shown at least once. Explain why:

a some of the roads will have to be swept twice

b the shortest distance the roadsweeper must travel is 57 units.

Find a route by which the roadsweeper can achieve this minimum.
24 The graph opposite shows the roads in Postman Peter’s mailing route. If the Post Office where Peter starts and finishes his round is at A, how should Peter minimise the distance he must walk?

25 A carnival procession wishes to march down each of the roads shown in the diagram given, in which all lengths are shown in kilometres.
   a List the three different ways in which the four odd vertices in the diagram can be paired.
   b Find the shortest distance that the procession has to travel if they are to start and finish at E.

26 Solve the Travelling Salesman Problem for the following networks given the salesman lives in village P. All distances are given in kilometres.

27 Find an adjacency matrix for:
   a
   b

28 Construct a network for the adjacency matrix:
   a
   b

29 Peter bought the following power tools for his construction business: 15 drills, 14 sanders, 8 bench saws, and 18 circular saws. Their individual costs were $45, $67, $315, and $56 respectively.
   a Write the item numbers as a $1 \times 4$ matrix $N$.
   b Write the costs as a $4 \times 1$ matrix $C$.
   c Use matrix methods only to find the total cost of the tools.
30 If \( A = \begin{bmatrix} 21 & 37 & 15 & 18 & 29 \\ 41 & 28 & 32 & 30 & 18 \\ 17 & 26 & 33 & 39 & 40 \end{bmatrix} \) and \( B = \begin{bmatrix} \vdots \\ 82 \\ 73 \\ 111 \\ 69 \\ 75 \end{bmatrix} \) find, if possible: \( a \ AB \)
\( b \ BA \)
\( c \ 32A \)

31 Calculate \( \begin{bmatrix} 21 \\ 7 \\ 18 \\ 32 \end{bmatrix} \)

32 A café sells two types of cola drinks. The drinks each come in three sizes: small, medium and large. At the beginning of the day the fridge was stocked with the number of units shown in the matrix below. At the end of the day the stock was again counted.

<table>
<thead>
<tr>
<th>Start of the day</th>
<th>At the end of the day</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brand C Brand P</td>
<td>Brand C Brand P</td>
</tr>
<tr>
<td>small</td>
<td>42 54</td>
</tr>
<tr>
<td>medium</td>
<td>36 27</td>
</tr>
<tr>
<td>large</td>
<td>34 30</td>
</tr>
<tr>
<td>small</td>
<td>27 31</td>
</tr>
<tr>
<td>medium</td>
<td>28 15</td>
</tr>
<tr>
<td>large</td>
<td>28 22</td>
</tr>
</tbody>
</table>

The profit matrix for each item is: small medium large

\[
\begin{bmatrix}
\$0.75 & \$0.55 & \$1.20
\end{bmatrix}
\]

Use matrix methods to calculate the total profit made for the day from the sale of these drinks.

33 Matrix \( A \) is \( 3 \times 7 \) and matrix \( B \) is \( n \times 4 \).

\( a \) When can matrix \( AB \) be calculated?
\( b \) If \( AB \) can be found, what is its order?
\( c \) Can \( BA \) be calculated?

34 Kelly has five women’s clothing shops: A, B, C, D and E. At these shops she sells standard items: skirts, dresses, suits and slacks. At the end of week 1 in May her stock holding is given by the matrix:

\[
\begin{bmatrix}
15 & 17 & 11 & 22 & 20 \\
28 & 31 & 26 & 10 & 30 \\
17 & 9 & 11 & 11 & 22 \\
13 & 32 & 30 & 27 & 8
\end{bmatrix}
\]

\( A \) \( B \) \( C \) \( D \) \( E \)

\[
\begin{bmatrix}
27 & 38 & 14 & 59 & 26 \\
31 & 42 & 29 & 16 & 35 \\
20 & 23 & 25 & 17 & 32 \\
26 & 59 & 40 & 31 & 17
\end{bmatrix}
\]

\( a \) Her sales matrix for week 2 was
\( b \) Her order matrix at the end of week 2 was
\( c \) If skirts cost her $38, dresses $75, suits $215 and slacks $48, how much will she have to pay for her end of week 2 order?

Find her stock matrix at the end of week 2.

Find her stock matrix at the start of week 3.